Lecture 10

Restricted Three-Body Problem

THUS far, only two bodies under the gravitational influence of each other have been considered. This lesson focuses on how to describe the motion associated with three bodies, given some initial conditions for position and velocity. The equilibrium solutions of the restricted three-body problem and their stability are also studied.

Overview

The general three-body problem has been shown by Poincaré and Bruns to lack an analytical solution given by algebraic terms and integrals. We consider a simplified version of the problem known as the "circular restricted three-body problem", that makes the following assumptions:

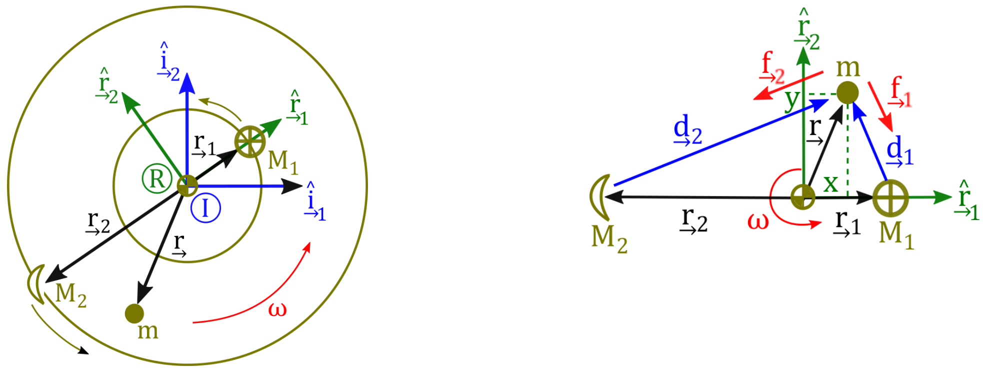
- Two large bodies with masses M_1 and M_2 are in circular orbits about their centre of mass.
- A third body with mass m, small enough to have negligible gravitational effects on M_1 and M_2, is subject to some initial position and velocity vectors, $\underset{\rightarrow}{\boldsymbol{r}}_0$ and $\underset{\rightarrow}{\boldsymbol{v}}_0$, such that it remains on the plane of motion of the other two.

The goal is to described the motion of m under these circumstances.

Equations of Motion

Consider a rotating frame, $\mathscr{F}_R$, with its origin at the centre of mass of M_1 and M_2, and its 1-axis pointing towards M_1. The mean motion of the system's rotation (such as that of the Earth-Moon system), or equivalently the angular rotation rate of $\mathscr{F}_R$ with respect to the inertial frame, $\mathscr{F}_I$ (also with its origin at the centre of mass, but with its basis vectors inertially fixed), both shown in Figure 10.1, is given by:

$$\omega = \sqrt{\frac{G(M_1 + M_2)}{(r_1 + r_2)^3}} \tag{10.1}$$



(a) Rotating in Inertial Frame

(b) Motion in Rotating Frame

Figure 10.1: Restricted Three-Body Problem, Studies in Inertial and Rotating Frames

The positions of M_1 and M_2 in $\mathscr{F}_R$ with respect to $O_R \equiv$ ◐ are given by $\underset{\to}{\boldsymbol{r}}_1$ and $\underset{\to}{\boldsymbol{r}}_2$ (fixed in $\mathscr{F}_R$). The positions and angular velocity vectors of interest can be resolved in $\mathscr{F}_R$ as follow:

$$\underset{\to}{\boldsymbol{r}} = \underset{\to}{\mathscr{F}}_R^{\mathsf{T}} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad , \quad \underset{\to}{\boldsymbol{\omega}} = \underset{\to}{\mathscr{F}}_R^{\mathsf{T}} \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \quad , \quad \underset{\to}{\boldsymbol{d}}_1 = \underset{\to}{\mathscr{F}}_R^{\mathsf{T}} \begin{bmatrix} x - r_1 \\ y \\ 0 \end{bmatrix} \quad , \quad \underset{\to}{\boldsymbol{d}}_2 = \underset{\to}{\mathscr{F}}_R^{\mathsf{T}} \begin{bmatrix} x + r_2 \\ y \\ 0 \end{bmatrix} \tag{10.2}$$

where $\underset{\to}{\boldsymbol{r}}$ is the position of m relative to O_R, $\underset{\to}{\boldsymbol{\omega}}$ is the system's angular velocity vector, and $\underset{\to}{\boldsymbol{d}}_1$ and $\underset{\to}{\boldsymbol{d}}_2$ denote the position of m relative to M_1 and M_2, respectively. The value of ω is given by Eq. (10.1), and $r_1 = |\underset{\to}{\boldsymbol{r}}_1|$ and $r_2 = |\underset{\to}{\boldsymbol{r}}_2|$ are constant.

The acceleration of m as measured in $\mathscr{F}_I$ can be obtained in two ways:

$$\underset{\to}{\boldsymbol{r}}^{\bullet\bullet} = \underset{\to}{\boldsymbol{r}}^{\circ\circ} + 2\underset{\to}{\boldsymbol{\omega}} \times \underset{\to}{\boldsymbol{r}}^{\circ} + \underset{\to}{\boldsymbol{\omega}}^{\circ} \times \underset{\to}{\boldsymbol{r}} + \underset{\to}{\boldsymbol{\omega}} \times (\underset{\to}{\boldsymbol{\omega}} \times \underset{\to}{\boldsymbol{r}}) = \underset{\to}{\mathscr{F}}_R^{\mathsf{T}}\Big(\overset{\bullet\bullet}{\boldsymbol{r}} + 2\boldsymbol{\omega}^{\times}\overset{\bullet}{\boldsymbol{r}} + \cancelto{\mathbf{0}}{\overset{\bullet}{\boldsymbol{\omega}}^{\times}}\boldsymbol{r} + \boldsymbol{\omega}^{\times}(\boldsymbol{\omega}^{\times}\boldsymbol{r})\Big) \tag{10.3a}$$

$$\underset{\to}{\boldsymbol{r}}^{\bullet\bullet} = \underset{\to}{\boldsymbol{f}}_1 + \underset{\to}{\boldsymbol{f}}_2 = -\frac{GM_1}{d_1^3}\underset{\to}{\boldsymbol{d}}_1 - \frac{GM_2}{d_2^3}\underset{\to}{\boldsymbol{d}}_2 \qquad = \underset{\to}{\mathscr{F}}_R^{\mathsf{T}}\Big(-\frac{GM_1}{d_1^3}\boldsymbol{d}_1 - \frac{GM_2}{d_2^3}\boldsymbol{d}_2\Big) \tag{10.3b}$$

among which Eq. (10.3a) is due to the transport theorem from KINEMATICS, while Eq. (10.3b) comes from Newton's law of gravitation from DYNAMICS. We have:

$$\overset{\bullet\bullet}{\boldsymbol{r}} = \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ 0 \end{bmatrix} \quad , \quad \boldsymbol{\omega}^{\times}\overset{\bullet}{\boldsymbol{r}} = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix} = \begin{bmatrix} -\omega\dot{y} \\ \omega\dot{x} \\ 0 \end{bmatrix} \quad , \quad \boldsymbol{\omega}^{\times}\boldsymbol{\omega}^{\times}\boldsymbol{r} = \begin{bmatrix} -\omega^2 x \\ -\omega^2 y \\ 0 \end{bmatrix} \tag{10.4}$$

using which while equating Eqs. (10.3a) and (10.3b) yields the equations of motion of m in the rotating system:

$$\ddot{x} - 2\omega\dot{y} - \omega^2 x = -\frac{GM_1}{d_1^3}(x - r_1) - \frac{GM_2}{d_2^3}(x + r_2) \triangleq f(x, y) \tag{10.5a}$$

$$\ddot{y} + 2\omega\dot{x} - \omega^2 y = -G\Big(\frac{M_1}{d_1^3} + \frac{M_2}{d_2^3}\Big)y \qquad \triangleq g(x, y) \tag{10.5b}$$

which is a 2$^{\text{nd}}$ order system of nonlinear ODE's that can be solved numerically for $x(t)$ and $y(t)$, given some initial conditions at $t = 0$: x_0, $\dot{x}_0$, y_0, and $\dot{y}_0$.

Lagrange Points

The equilibrium solutions to the equations of motion in Eq. (10.5) give rise to the so-called “Lagrange points”. Objects placed at these points remain stationary in the rotating frame, $\mathscr{F}_R$, as long as they are not perturbed. Letting $\dot{x} = \ddot{x} = 0$ and $\dot{y} = \ddot{y} = 0$ in Eq. (10.5) and rearranging yields:

$$\omega^2 x_e = G\Big[\frac{M_1}{d_1^3}(x_e - r_1) + \frac{M_2}{d_2^3}(x_e + r_2)\Big] \tag{10.6a}$$

$$\Big[G\Big(\frac{M_1}{d_1^3} + \frac{M_2}{d_2^3}\Big) - \omega^2\Big] y_e = 0 \tag{10.6b}$$

where the subscript ’e’ denotes ’equilibrium’. Two sets of solutions are obtained based on Eq. (10.6b):

i) If $y_e = 0$, the geometry of Figure 10.1b implies:

$$d_1 = |x_e - r_1| \;\;,\;\; d_2 = |x_e + r_2| \tag{10.7}$$

substituting which into Eq. (10.6a) yields:

$$\omega^2 x_e = G\Big[\frac{M_1}{|x_e - r_1|^3}(x_e - r_1) + \frac{M_2}{|x_e + r_2|^3}(x_e + r_2)\Big] \tag{10.8}$$

which is now an equation in x_e only, and has 3 real roots. These roots provide the position, on the 1-axis of $\mathscr{F}_R$ (on which both M_1 and M_2 lie), of three of the Lagrange points, called $\mathscr{L}_1$, $\mathscr{L}_2$, and $\mathscr{L}_3$, and shown in Figure 10.2a.

ii) If $y_e \neq 0$, Eq. (10.6b) implies:

$$\omega^2 = G\Big(\frac{M_1}{d_1^3} + \frac{M_2}{d_2^3}\Big) \tag{10.9}$$

substituting which into Eq. (10.6a) yields:

$$G\Big(\frac{M_1}{d_1^3}x_e + \frac{M_2}{d_2^3}x_e\Big) = G\Big[\frac{M_1}{d_1^3}(x_e - r_1) + \frac{M_2}{d_2^3}(x_e + r_2)\Big] \;\;\Rightarrow\;\; 0 = -\cancel{G}\frac{M_1 r_1}{d_1^3} + \cancel{G}\frac{M_2 r_2}{d_2^3} \tag{10.10}$$

But by definition of the centre of mass (from DYNAMICS) that coincides with O_R, we have $M_1 r_1 = M_2 r_2$, which simplifies Eq. (10.10) to:

$$d_1^3 = d_2^3 \;\;\Rightarrow d_1 = d_2 = d \tag{10.11}$$

substituting which back into Eq. (10.9) while recalling the value of ω from Eq. (10.1) yields:

$$\omega^2 = \left(\sqrt{G\frac{M_1 + M_2}{(r_1 + r_2)^3}}\right)^2 = \frac{G(M_1 + M_2)}{d^3} \;\;\Rightarrow\;\; d_1 = d_2 = d = r_1 + r_2 \tag{10.12}$$

which provide the position of the remaining two Lagrange points in $\mathscr{F}_R$, called $\mathscr{L}_4$ and $\mathscr{L}_5$, and shown in Figure 10.2a. They are arranged in the form of two equilateral triangles, with M_1 and M_2 at the endpoints of their common base.

Stability of Lagrange Points

It can be shown that the Lagrange points $\mathscr{L}_1$, $\mathscr{L}_2$, and $\mathscr{L}_3$ are unstable; while $\mathscr{L}_4$ and $\mathscr{L}_5$ are stable if $M_1/M_2 < 0.04$ or $M_2/M_1 < 0.04$, and they are unstable otherwise. This means small motions of an object in the vicinity of the last two Lagrange points will remain that way, without growing boundlessly.

Let $x = x_e + \delta x$, with $\dot{x}_e = \ddot{x}_e = 0$; and $y = y_e + \delta y$, with $\dot{y}_e = \ddot{y}_e = 0$. The perturbations $\delta x(t)$ and $\delta y(t)$ are assumed to be small. Substituting these forms into Eq. (10.5) and using Taylor series to linearize the functions yields:

$$\delta\ddot{x} - 2\omega\delta\dot{y} - \omega^2\delta x - \omega^2 x_e = f_0 + \cancelto{\triangleq A}{\frac{\partial f}{\partial x}\Big|_{x_e,y_e}}\delta x + \cancelto{\triangleq B}{\frac{\partial f}{\partial y}\Big|_{x_e,y_e}}\delta y \tag{10.13a}$$

$$\delta\ddot{y} + 2\omega\delta\dot{x} - \omega^2\delta y - \omega^2 y_e = g_0 + \cancelto{\triangleq C}{\frac{\partial g}{\partial x}\Big|_{x_e,y_e}}\delta x + \cancelto{\triangleq D}{\frac{\partial g}{\partial y}\Big|_{x_e,y_e}}\delta y \tag{10.13b}$$

where $f_0 \triangleq f(x_e, y_e)$ and $g_0 \triangleq g(x_e, y_e)$, but from Eq. (10.6), we also have $-\omega^2 x_e = f(x_e, y_e) = f_0$ and $-\omega^2 y_e = g(x_e, y_e) = g_0$, substituting which into Eq. (10.13) and rearranging yields:

$$\delta\ddot{x} - 2\omega\delta\dot{y} - (\omega^2 + A)\delta x - B\delta y + \cancel{f_0} = \cancel{f_0} \tag{10.14a}$$

$$\delta\ddot{y} + 2\omega\delta\dot{x} - (\omega^2 + D)\delta y - C\delta x + \cancel{g_0} = \cancel{g_0} \tag{10.14b}$$

which is a system of 2$^{\text{nd}}$ order linear constant-coefficient ODE's. Taking solutions of the form $\delta x(t) = \bar{x}e^{\lambda t}$ and $\delta y(t) = \bar{y}e^{\lambda t}$, the system in Eq. (10.14) can be written in the following matrix form:

$$e^{\lambda t}\begin{bmatrix} \lambda^2 - \omega^2 - A & -2\omega\lambda - B \\ 2\omega\lambda - C & \lambda^2 - \omega^2 - D \end{bmatrix}\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{10.15}$$

For a non-trivial solution, the determinant of the 2×2 matrix in Eq. (10.15) should be zero. This requirement provides a 4$^{\text{th}}$ order equation in λ (with different sets of coefficients of each Lagrange point) that has 4 roots, using which the associated Lagrange point's stability can be determined:

- *stable*: if all λ_i's have $\mathrm{Re}(\lambda_\mathrm{i}) < 0$, or $\mathrm{Re}(\lambda_\mathrm{i}) = 0$ for distinct λ_i's ($\lambda_i \neq 0$)
- *unstable*: if at least one λ_i has $\mathrm{Re}(\lambda_\mathrm{i}) > 0$

These conditions come from considering the boundedness of $e^{\lambda_i t}$.

A Conserved Quantity

Analogously to the specific orbital energy and the other constants in a two-body motion, the restricted three-body motion also has a conserved quantity.

Definition 1. The *Jacobi integral* is a constant of motion associated with the circular restricted three-body problem:

$$C_J \triangleq \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \Big[G\Big(\frac{M_1}{d_1} + \frac{M_2}{d_2}\Big) + \frac{\omega^2}{2}(x^2 + y^2)\Big] \tag{10.16}$$

the value of which is determined by the system's initial conditions. The term outside the square brackets represents kinetic energy *in the rotating frame* (hence not true) of the body, the first term inside the brackets describes its true potential energy, and the last term relates to the contribution of the centrifugal force.

To see where this quantity comes from, we revisit the nonlinear equations of motion in Eq. (10.5): multiplying Eq. (10.5a) by $\dot{x}$ and Eq. (10.5b) by $\dot{y}$ and adding the results yields:

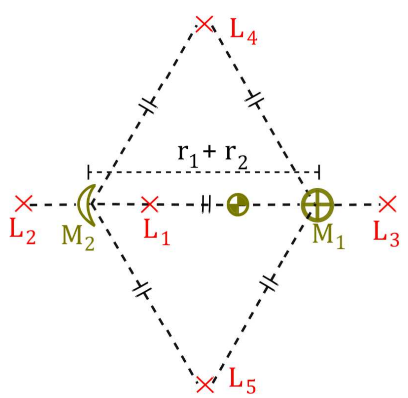
$$\dot{x}\ddot{x} + \dot{y}\ddot{y} - \omega^2(x\dot{x} + y\dot{y}) = -G\Big[\frac{M_1}{d_1^3}(x - r_1)\dot{x} + \frac{M_2}{d_2^3}(x + r_2)\dot{x} + \Big(\frac{M_1}{d_1^3} + \frac{M_2}{d_2^3}\Big)y\dot{y}\Big] \tag{10.17}$$

which should be recognized as a rearranged form of the derivative of our conserved quantity:

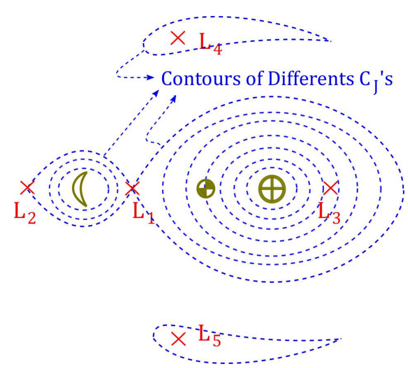
$$\frac{d}{dt}\Big[\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - G\Big(\frac{M_1}{d_1} + \frac{M_2}{d_2}\Big) - \frac{\omega^2}{2}(x^2 + y^2)\Big] = 0 \tag{10.18}$$

where $d(d_1^2)/dt = 2d_1\dot{d}_1 = d\big[(x - r_1)^2 + y^2\big]/dt$ and $d(d_2^2)/dt = 2d_2\dot{d}_2 = d\big[(x + r_2)^2 + y^2\big]/dt$ - based on Figure 10.1b - and Eq. (10.2) are used.

The contour plots of Jacobi's integral equation are known as "Hill curves". As an example of how these contours can be used for analysis, we consider the zero relative velocity curves, shown in Figure 10.2b. These curves are obtained when $\dot{x} = \dot{y} = 0$ in Eq. (10.16), and show the regions of possible motion about each primary, in order to move beyond which the spacecraft needs thrust. Large negative values of C_J imply either the spacecraft can be far from the centre of mass (large $x^2 + y^2$), or it can be in the vicinity of one of the two large bodies (small d_1 or small d_2). The difference of the C_J values between two periodic orbits provides an estimate of how much thrust is required to transfer between them.



(a) Earth-Moon System's Lagrange Points



(b) Hill Curves for Zero Relative Velocity

Figure 10.2: Arrangement of Lagrange Points, and Illustration of Hill Curves