Lecture 15

Energy Dissipation in Dual-Spins

THE effects of internal energy dissipation on stability of dual-spin vehicles are studied. The previously-encountered energy sink hypothesis is applied, similarly to the dissipation-free stability analysis, to the two cases involving a nominally non-spinning platform and a wheel aligned with one of the vehicle's principal axes. A constrained minimization approach is used to find the angular velocity corresponding to the system's minimum kinetic energy, and the stability conditions are derived.

Overview

Recall, from SPIN STABILIZATION, the definition of a "quasi-rigid" body with slow internal dissipation of its kinetic energy. Considering that a dual-spin consists of two main components, we focus on a gyrostat with a *quasi-rigid carrier* and a *rigid wheel.* A body-fixed principal axes frame, $\mathscr{F}_P$ (corresponding to the system's overall principal moment of inertia matrix), is considered. The total angular velocity of the system, $\underrightarrow{\omega} = \underrightarrow{\mathscr{F}}_P^{\mathsf{T}}\boldsymbol{\omega}$, is arbitrarily general.

Note: Refer to Section 7.1 of *Spacecraft Attitude Dynamics* for a more detailed study of internal energy dissipation in multi-spin vehicles.

Let $\boldsymbol{h}_s \triangleq I_s\omega_s\hat{\boldsymbol{a}}$ represent the angular momentum of the wheel as expressed in $\mathscr{F}_P$, with I_s representing the wheel's moment of inertia about the spin axis, ω_s denoting its rotation rate, and $\hat{\boldsymbol{a}}$ representing its spin axis as expressed in $\mathscr{F}_P$. Recall the following quantities considered in DUAL-SPIN STABILIZATION:

$$T_p = \frac{1}{2}\boldsymbol{\omega}^{\mathsf{T}}\boldsymbol{I}\boldsymbol{\omega} = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) \tag{15.1a}$$

$$h^2 = \boldsymbol{h}^{\mathsf{T}}\boldsymbol{h} \qquad = (\boldsymbol{I}\boldsymbol{\omega} + \boldsymbol{h}_s)^{\mathsf{T}}(\boldsymbol{I}\boldsymbol{\omega} + \boldsymbol{h}_s) \tag{15.1b}$$

both of which were shown to be constant for a rigid carrier with a rigid wheel. It should be noted that $\boldsymbol{\omega}$, $\boldsymbol{h}$, and $\boldsymbol{I}$ are all expressed in $\mathscr{F}_P$. As a result of internal energy dissipation, the energy sink hypothesis postulates that T_p should *slowly* decrease until it reaches a minimum value, while h is assumed to remain constant. The following constrained optimization problem is, thus, formulated:

$$\text{Objective} : \min_{\boldsymbol{\omega}}\left\{\boldsymbol{\omega}^{\mathsf{T}}\boldsymbol{I}\boldsymbol{\omega} - \frac{(\boldsymbol{I}\boldsymbol{\omega} + \boldsymbol{h}_{\mathrm{s}})^{\mathsf{T}}(\boldsymbol{I}\boldsymbol{\omega} + \boldsymbol{h}_{\mathrm{s}})}{\lambda}\right\} \tag{15.2}$$

where λ is a Lagrange multiplier that is used to adjoin the original objective function, $2T_p = \boldsymbol{\omega}^{\mathsf{T}}\boldsymbol{I}\boldsymbol{\omega}$, with the equality constraint $(\boldsymbol{I}\boldsymbol{\omega}+\boldsymbol{h}_s)^{\mathsf{T}}(\boldsymbol{I}\boldsymbol{\omega}+\boldsymbol{h}_s) = h^2 = \text{constant}$. The stationary points corresponding to the local minima or maxima are obtained by:

$$\frac{\partial}{\partial \boldsymbol{\omega}}\Big(2T_p - \frac{h^2}{\lambda}\Big) = 2\boldsymbol{I}\Big(\boldsymbol{\omega} - \frac{\boldsymbol{I}\boldsymbol{\omega}+\boldsymbol{h}_s}{\lambda}\Big) = \boldsymbol{0} \tag{15.3}$$

from which one can conclude that the extrema of T_p subject to the constant h constraint occur when:

$$\boldsymbol{\omega}_0 = \frac{\boldsymbol{I}\boldsymbol{\omega}_0+\boldsymbol{h}_s}{\lambda} \quad\Rightarrow\quad \boldsymbol{h}_0 = \lambda\boldsymbol{\omega}_0 \quad\Rightarrow\quad \boldsymbol{\omega}_0^{\times}(\boldsymbol{I}\boldsymbol{\omega}_0+\boldsymbol{h}_s) = \boldsymbol{0} \tag{15.4}$$

which is equivalent to the equilibrium condition encountered in DUAL-SPIN STABILIZATION. To check whether the extremum of interest is a local minimum or a local maximum, the Hessian of T_p is required.

Let $\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \delta\boldsymbol{\omega}$. We then have:

$$\Delta T_{p_0} \triangleq T_p - T_{p_0} = \frac{1}{2}\boldsymbol{\omega}^{\mathsf{T}}\boldsymbol{I}\boldsymbol{\omega} - \frac{1}{2}\boldsymbol{\omega}_0^{\mathsf{T}}\boldsymbol{I}\boldsymbol{\omega}_0 = \boldsymbol{\omega}_0^{\mathsf{T}}\boldsymbol{I}\delta\boldsymbol{\omega} + \frac{1}{2}\delta\boldsymbol{\omega}^{\mathsf{T}}\boldsymbol{I}\delta\boldsymbol{\omega} \tag{15.5}$$

but the problem's constraint requires that $|\boldsymbol{h}|^2 = |\boldsymbol{h}_0|^2 = h^2$, which expands as :

$$[\boldsymbol{I}(\boldsymbol{\omega}_0+\delta\boldsymbol{\omega})+\boldsymbol{h}_s]^{\mathsf{T}}[\boldsymbol{I}(\boldsymbol{\omega}_0+\delta\boldsymbol{\omega})+\boldsymbol{h}_s] = (\boldsymbol{I}\boldsymbol{\omega}_0+\boldsymbol{h}_s)^{\mathsf{T}}(\boldsymbol{I}\boldsymbol{\omega}_0+\boldsymbol{h}_s) \tag{15.6}$$

simplifying which further yields:

$$\boldsymbol{h}^{\mathsf{T}}\boldsymbol{h} - \boldsymbol{h}_0^{\mathsf{T}}\boldsymbol{h}_0 = 2\delta\boldsymbol{\omega}^{\mathsf{T}}\boldsymbol{I}\overset{=\boldsymbol{h}_0=\lambda\boldsymbol{\omega}_0}{\cancel{(\boldsymbol{I}\boldsymbol{\omega}_0+\boldsymbol{h}_s)}} + \delta\boldsymbol{\omega}^{\mathsf{T}}\boldsymbol{I}^2\delta\boldsymbol{\omega} = 0 \quad\Rightarrow\quad 2\lambda\boldsymbol{\omega}_0^{\mathsf{T}}\boldsymbol{I}\delta\boldsymbol{\omega} = -\delta\boldsymbol{\omega}^{\mathsf{T}}\boldsymbol{I}^2\delta\boldsymbol{\omega} \approx 0 \tag{15.7}$$

where $\boldsymbol{\omega}_0^{\mathsf{T}}\boldsymbol{I}\delta\boldsymbol{\omega} \approx 0$ may be neglected to obtain a first order approximation. We can also use Eq. (15.7) to express one of $\delta\boldsymbol{\omega}$'s components in terms of the other two, which will later be treated as the two independent variables of the problem:

$$\omega_{0_1}I_1\delta\omega_1 + \omega_{0_2}I_2\delta\omega_2 + \omega_{0_3}I_3\delta\omega_3 = 0 \quad\Rightarrow\quad \delta\omega_3 = -\frac{\omega_{0_1}I_1\delta\omega_1 + \omega_{0_2}I_2\delta\omega_2}{\omega_{0_3}I_3} \tag{15.8}$$

Now, pre-multiplying Eq. (15.5) by 2λ and employing the right-hand side result in Eq. (15.7) yields:

$$2\lambda\Delta T_{p_0} = \overset{-\delta\boldsymbol{\omega}^{\mathsf{T}}\boldsymbol{I}^2\delta\boldsymbol{\omega}}{\cancel{2\lambda\boldsymbol{\omega}_0^{\mathsf{T}}\boldsymbol{I}\delta\boldsymbol{\omega}}} + \lambda\delta\boldsymbol{\omega}^{\mathsf{T}}\boldsymbol{I}\delta\boldsymbol{\omega} = \delta\boldsymbol{\omega}^{\mathsf{T}}\boldsymbol{I}(\lambda\boldsymbol{1}-\boldsymbol{I})\delta\boldsymbol{\omega} \tag{15.9}$$

which can be expanded in component form and written, using Eq. (15.8), as:

$$2\lambda\Delta T_{p_0} = \delta\omega_1^2 I_1(\lambda - I_1) + \delta\omega_2^2 I_2(\lambda - I_2) + \Big(\frac{\omega_{0_1}I_1\delta\omega_1 + \omega_{0_2}I_2\delta\omega_2}{\omega_{0_3}I_3}\Big)^2 I_3(\lambda - I_3) \tag{15.10}$$

which can, finally, be further expanded and recast in the following form:

$$\begin{bmatrix}\delta\omega_1 & \delta\omega_2\end{bmatrix}\boldsymbol{H}\begin{bmatrix}\delta\omega_1\\ \delta\omega_2\end{bmatrix} = \Delta T_{p0} \tag{15.11}$$

where it is evident that $\Delta T_{p_0} > 0$ corresponding to a local minimum of T_{p_0} requires a positive-definite

Hessian, $\boldsymbol{H}$. The following conditions should be satisfied in order for $\boldsymbol{H}$ to possess this property:

$$0 < \lambda\big[I_1(\lambda - I_2)\omega_{0_1}^2 + I_2(\lambda - I_1)\omega_{0_2}^2\big] \tag{15.12a}$$
$$0 < \lambda\big[I_2(\lambda - I_3)\omega_{0_2}^2 + I_3(\lambda - I_2)\omega_{0_3}^2\big] \tag{15.12b}$$
$$0 < \lambda\big[I_3(\lambda - I_1)\omega_{0_3}^2 + I_1(\lambda - I_3)\omega_{0_1}^2\big] \tag{15.12c}$$
$$0 < I_1(\lambda - I_2)(\lambda - I_3)\omega_{0_1}^2 + I_2(\lambda - I_3)(\lambda - I_1)\omega_{0_2}^2 + I_3(\lambda - I_1)(\lambda - I_2)\omega_{0_3}^2 \tag{15.12d}$$

The condition in Eq. (15.12d), together with one of the conditions among Eqs. (15.12a), (15.12b), and (15.12c) satisfies the other two, leading to the "major axis rule" (presented earlier in SPIN STABILIZATION for gyrostats with platform energy dissipation). Similar - but less general - conditions can be derived using the same approach for the two special cases considered in DUAL-SPIN STABILIZATION.

Nominally Non-Spinning Platform

Following a similar approach to the more general one presented thus far and used to obtain Eq. (15.12) (since they cannot be directly applied here for $\lambda \to \infty$), and letting $\boldsymbol{\omega}_0 = \mathbf{0}$ and $\boldsymbol{\omega} = \delta\boldsymbol{\omega}$, we have:

$$T_{p_0} = \frac{1}{2}\boldsymbol{\omega}_0^\intercal \boldsymbol{I}\boldsymbol{\omega}_0 = 0 \;\;,\;\; \Delta T_{p_0} = \cancelto{\mathbf{0}}{\boldsymbol{\omega}_0^\intercal \boldsymbol{I}\delta\boldsymbol{\omega}} + \frac{1}{2}\delta\boldsymbol{\omega}^\intercal \boldsymbol{I}\delta\boldsymbol{\omega} = \frac{1}{2}\delta\boldsymbol{\omega}^\intercal \boldsymbol{I}\delta\boldsymbol{\omega} \tag{15.13}$$

where Eq. (15.5) is used. The energy change ΔT_{p_0} is shown to be positive, owing to the moment of inertia matrix's positive-definiteness, even before the constant h^2 constraint is imposed. This means any motion of the quasi-rigid carrier in this nominally non-spinning case increases the T_p energy of interest.

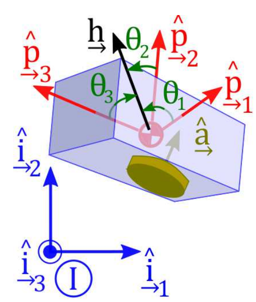

Figure 15.1: Direction Cosines

The total angular momentum can be resolved in frame $\mathscr{F}_P$ in terms of its direction cosines:

$$\boldsymbol{h} = \boldsymbol{I}\boldsymbol{\omega} + \boldsymbol{h}_s = \boldsymbol{I}\delta\boldsymbol{\omega} + \boldsymbol{h}_s \;\;\Rightarrow\;\; \boldsymbol{I}\delta\boldsymbol{\omega} + h_s\hat{\boldsymbol{a}} = h\hat{\boldsymbol{c}} \;\;,\;\; \hat{\boldsymbol{c}} \triangleq \begin{bmatrix}\cos(\theta_1)\\ \cos(\theta_2)\\ \cos(\theta_3)\end{bmatrix} \tag{15.14}$$

where θ_i represents the angle from $\hat{\underset{\rightarrow}{\boldsymbol{p}}}_i$ to $\underset{\rightarrow}{\boldsymbol{h}}$, shown in Figure 15.1. The constant h assumption is evoked, leading to $h \approx h_s$ in this case. Solving Eq. (15.14) for $\delta\boldsymbol{\omega}$ and substituting the result into Eq. (15.13) yields:

$$\Delta T_{p_0} = \frac{h^2}{2}\Big[\frac{(c_1 - a_1)^2}{I_1} + \frac{(c_2 - a_2)^2}{I_2} + \frac{(c_3 - a_3)^2}{I_3}\Big] \geqslant 0 \tag{15.15}$$

for which to be 0, in accordance with our energy sink hypothesis, we need $c_i = a_i$ for all $i \in \{1, 2, 3\}$. The conclusion is that a gyrostat with a nominally non-spinning quasi-rigid platform will have its body-fixed wheel spin axis, $\hat{\underset{\rightarrow}{\boldsymbol{a}}}$, asymptotically approaching $\underset{\rightarrow}{\boldsymbol{h}}$.

Wheel Aligned with System Principal Axis

As considered in DUAL-SPIN STABILIZATION, in this case a nominal angular velocity vector along one of the overall principal axes of the body (assumed, without loss of generality) to be the 2-axis, $\underset{\rightarrow}{\hat{\boldsymbol{p}}}_2$) is achieved:

$$\boldsymbol{\omega}_0 = \begin{bmatrix} 0 \\ \nu \\ 0 \end{bmatrix} \quad , \quad \boldsymbol{\omega} = \boldsymbol{\omega}_0 + \delta\boldsymbol{\omega} = \begin{bmatrix} \delta\omega_1 \\ \nu + \delta\omega_2 \\ \delta\omega_3 \end{bmatrix} \tag{15.16}$$

Following a similar approach to that used to obtain the conditions in Eq. (15.12) (since they cannot be directly applied owing to their use of Eq. (15.8) which is rendered invalid with $\omega_{0_3} = 0$), the following conditions for positive-definiteness of $\boldsymbol{H}$ are arrived at:

$$0 < \lambda I_2(\lambda - I_1)\nu^2 \quad \Rightarrow \quad \lambda(\lambda - I_1) > 0 \tag{15.17a}$$

$$0 < \lambda I_2(\lambda - I_3)\nu^2 \quad \Rightarrow \quad \lambda(\lambda - I_3) > 0 \tag{15.17b}$$

where $\lambda = I_2 + h_s/\nu$ from left-hand side of Eq. (15.4). These conditions can be broken down as follows:

$$\lambda > I_1 \text{ and } \lambda > \mathrm{I}_3 \ \ \textit{or} \ \ \lambda < 0 \tag{15.18}$$

The left-hand side set of conditions in Eq. (15.18) implies $I_2 + h_s/\nu$, the system's "modified" inertia owing to the wheel's rotation, should be larger than the two principal inertias. The right-hand side condition is a more restrictive analogue of the $\lambda < I_1$ and $\lambda < I_3$ condition seen before for a gyrostat with a rigid platform. Therefore, this particular type of gyrostat can stabilize both its minor and intermediate axes with a suitable choice of h_s. Unlike DUAL-SPIN STABILIZATION, however, satisfying the conditions in Eq. (15.18) will now guarantee *asymptotic $\boldsymbol{\omega}$-stability*, since $\boldsymbol{\omega}(t) \rightarrow \boldsymbol{\omega}_0$ as $t \rightarrow \infty$. The previously stable (with a rigid platform) case of $\lambda < I_1$ and $\lambda < I_3$ (but both non-negative) will now result in *instability*. A sample $k_1 - k_3$ stability diagram, with $k_1 \triangleq (I_2 - I_3)/I_1$ and $k_3 \triangleq (I_2 - I_1)/I_3$, and using $\hat{\Omega}_{p_0} \triangleq h_s/(\nu\sqrt{I_1 I_3}) = 0.5$ is shown in Figure 15.2, and should be compared against Figure 14.4 of DUAL-SPIN STABILIZATION that uses the same $\hat{\Omega}_{p_0}$ but disregards energy dissipation.

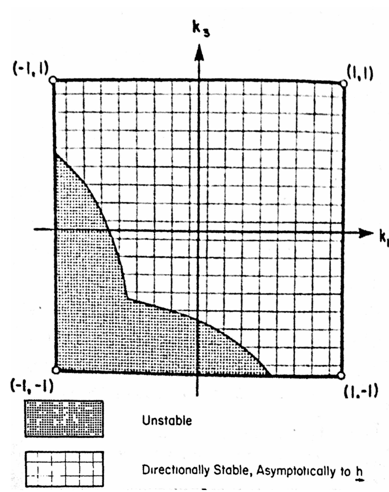

Figure 15.2: Dual-Spin $\hat{\Omega}_{po} = 0.5$ with Energy Dissipation [Hughes] (used with permission)

Note: Recall, from STABILITY, the consequences of adding a damping term to a conservative gyric system: if it was *statically stable*, it becomes *asymptotically stable*; if it was *gyrically stable*, it becomes *unstable*.

References

[Hughes] Hughes, P. C., *Spacecraft Attitude Dynamics*, Dover Publications Inc., New York, Chap. 7.