Lecture 16

Disturbance Torques

PROCEEDING forward from torque-free motion, external torques that affect the attitude motion of spacecraft are studied in this lesson. Several major sources of disturbance torques are considered, and mathematical models are provided to assist with estimating such torques.

Overview

Although disturbing torques due to the spacecraft's environment are, in an absolute sense, "small", their presence still has a noticeable influence owing to the lack of other major forces, such as large gravitational forces. The following major sources of disturbance torques are considered:

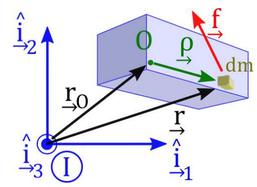

Figure 16.1: Overview

- magnetic field: magnetic and Lorentz forces on moving dipoles and charges
- aerodynamic forces: lift/drag forces from residual atmospheric particles
- solar radiation pressure: from photons' transfer of momentum upon "impact" on spacecraft's surface
- gravity gradient: from a varying spectrum of gravitational forces on different parts of spacecraft

The following relationship from DYNAMICS will be used to model the torque, $\vec{\boldsymbol{\tau}}$, on a rigid body and resulting from each of the above forces, $\vec{\boldsymbol{f}}$, as depicted in Figure 16.1:

$$\vec{\boldsymbol{\tau}} = \iiint_V \vec{\boldsymbol{\rho}} \times \vec{\boldsymbol{f}}(\vec{\boldsymbol{\rho}})\, dV \tag{16.1}$$

where $\vec{\boldsymbol{\rho}}$ represents the position of a differential mass element, $dm = \sigma(\vec{\boldsymbol{\rho}})\, dV$ (with σ denoting volumetric mass density), relative to a body-fixed reference point, taken to be $O \equiv$ ◕.

Magnetic Torque

This source is particularly noticeable for near-Earth spacecraft that experience larger geomagnetic field, which falls off proportionally to $1/r^3$, where r is the distance from Earth's centre. External torque is applied on the spacecraft as a result of the interaction of its magnetization (accumulated from electric currents running through the on-board electronics or magnetic torquers, or from permanent magnets) with Earth's magnetic field. We have:

$$\underset{\rightarrow}{\tau}_{mag} = \underset{\rightarrow}{m} \times \underset{\rightarrow}{b} \tag{16.2}$$

where $\underset{\rightarrow}{m}$ is the spacecraft's net magnetic dipole moment and $\underset{\rightarrow}{b}$ denotes the geomagnetic field vector, subject to both spatial and temporal changes. For current coils, such as those present in magnetic torquers (used for attitude control purposes), $\underset{\rightarrow}{m}(t) = i(t)A\underset{\rightarrow}{\hat{n}}$, where i is the electric current, A is the coil's cross-sectional area, and $\underset{\rightarrow}{\hat{n}}$ represents the unit normal to the cross-section.

Note: If the spacecraft's $\underset{\rightarrow}{m} = \underset{\rightarrow}{0}$, it remains $\underset{\rightarrow}{0}$ for all attitudes, unless it is actively modified to control the attitude.

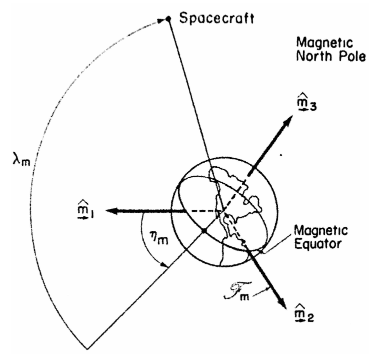

Figure 16.2: Latitude [Hughes]
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Several models that attempt to describe the geomagnetic field have been proposed. The "tilted dipole" model is a simple one that considers Earth as a large tilted magnet, and associates some magnetic potential with any spatial point in its vicinity:

$$\Phi_m = -\frac{\mu_m}{r^2}\sin(\lambda_m) \tag{16.3}$$

where r and λ_m are the point's distance from Earth's centre and latitude with respect to the geomagnetic equator, shown in Figure 16.2, and $\mu_m = 10^{17}$ Wb · m is Earth's dipole strength.

The International Geomagnetic Reference Field (IGRF) "spherical harmonics" model of Earth's magnetic field fits certain functions, namely the spherical harmonics encountered in ORBITAL PERTURBATIONS, to a series of past measurements, hence constructing an observation-based statistical model. This approach determines the field potential to k^{th} degree as follows:

$$\Phi_m = R_\oplus \sum_{n=1}^{k} \left(\frac{R_\oplus}{r}\right)^{n+1} \sum_{m=0}^{n} \Big[g_n^m \cos(m\eta) + h_n^m \sin(m\eta)\Big] P_n^m(\phi) \tag{16.4}$$

where P_n^m are Schmidt-normalized Legendre functions, and h_n^m and g_n^m are the so-called Gaussian coefficients of degree n and order m, obtained from periodically-updated IGRF tables. Earth's average radius is denoted by $R_\oplus$, and the spherical coordinates r, ϕ, and η represent distance, coelevation ($\phi = 90° - \lambda$), and East longitude from Greenwich (which itself has a longitude of η_G), respectively, and are depicted in Figure 16.3.

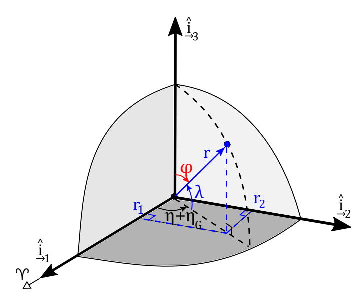

Figure 16.3: Coordinates

Once Φ_m is obtained using one of the models such as those above, the magnetic field vector can be calculated using $\underset{\rightarrow}{b} = -\underset{\rightarrow}{\nabla}\Phi_m$. For a $k = 1$ degree spherical harmonics model (that collapses down to the tilted dipole model), the following spherical components of $\underset{\rightarrow}{\boldsymbol{b}}$ are obtained:

$$b_r = -\frac{\partial \Phi}{\partial r} = 2\left(\frac{R_\oplus}{r}\right)^3 \left[g_1^0 \cos(\phi) + \left(g_1^1 \cos(\eta) + h_1^1 \sin(\eta)\right) \sin(\phi)\right] \tag{16.5a}$$

$$b_\phi = -\frac{1}{r}\frac{\partial \Phi}{\partial \phi} = \left(\frac{R_\oplus}{r}\right)^3 \left[g_1^0 \sin(\phi) - \left(g_1^1 \cos(\eta) + h_1^1 \sin(\eta)\right) \cos(\phi)\right] \tag{16.5b}$$

$$b_\eta = -\frac{1}{r\sin(\phi)}\frac{\partial \Phi}{\partial \eta} = \left(\frac{R_\oplus}{r}\right)^3 \left[g_1^1 \sin(\eta) - h_1^1 \cos(\eta)\right] \tag{16.5c}$$

where, based on the 2015 values of the 12$^{\text{th}}$ generation IGRF, $g_1^0 = -29442.0$ nT, $g_1^1 = -1501.0$ nT, and $h_1^1 = 4797.1$ nT. The IGRF table also includes predictions of secular variations, reported in nT/yr from 2015 to 2020, associated with each coefficient.

Aerodynamic Torque

Torques are exerted on spacecraft as a result of aerodynamic forces imparted by the residual atmosphere, but since the particles' density is very low, continuum fluid mechanics can no longer be applied. Since this type of disturbance is proportional to atmospheric density, it falls off exponentially with distance and becomes negligibly small beyond $r = 10^3$ km from Earth's centre.

The following assumptions are made in this section in order to model aerodynamics torques:

- atmospheric thermal motion much smaller than spacecraft's speed, implying $\underset{\rightarrow}{\boldsymbol{v}} \approx \mathbf{0}$ before impact
- complete loss of momentum of arriving molecules, resulting in $\underset{\rightarrow}{\boldsymbol{v}} = \underset{\rightarrow}{\boldsymbol{v}}_{orbit}$ upon impact
- nominally non-spinning spacecraft

Consider a streamtube of length $v\delta t$, as shown in Figure 16.4, through which a mass of δm passes over time δt. We have:

$$\delta m = \sigma_a v \delta t \; dA = \sigma_a v \delta t (\hat{\underset{\rightarrow}{\boldsymbol{v}}} \cdot \hat{\underset{\rightarrow}{\boldsymbol{n}}}) \; dS \quad , \quad \hat{\underset{\rightarrow}{\boldsymbol{v}}} \cdot \hat{\underset{\rightarrow}{\boldsymbol{n}}} = \cos(\alpha_a) \tag{16.6}$$

where σ_a represents atmospheric density and dS is a differential surface element (with outward unit normal, $\hat{\underset{\rightarrow}{\boldsymbol{n}}}$) projected onto dA, normal to $\hat{\underset{\rightarrow}{\boldsymbol{v}}}$. There is an angle of α_a between $\hat{\underset{\rightarrow}{\boldsymbol{n}}}$ and $\underset{\rightarrow}{\boldsymbol{v}}$, which changes as the direction of "flow" changes. The linear momentum of the particles before and after surface interaction can be expressed as:

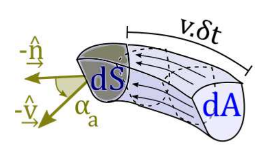

Figure 16.4: Streamtube

$$\underset{\rightarrow}{\boldsymbol{p}}(t) = \delta m \underset{\rightarrow}{\mathbf{0}} = \underset{\rightarrow}{\mathbf{0}} \quad , \quad \underset{\rightarrow}{\boldsymbol{p}}(t+\delta t) = \delta m \underset{\rightarrow}{\boldsymbol{v}} = \sigma_a v^2 \delta t \cos(\alpha_a) dS \frac{\underset{\rightarrow}{\boldsymbol{v}}}{v} \tag{16.7}$$

using which with $\underset{\rightarrow}{\boldsymbol{p}}^\bullet = \underset{\rightarrow}{\boldsymbol{f}}$, the differential force on the surface due to the impending air molecules, illustrated in Figure (16.5), can be determined:

$$\frac{d\underset{\rightarrow}{\boldsymbol{p}}}{dt} = \lim_{\delta t \to 0} \frac{\underset{\rightarrow}{\boldsymbol{p}}(t+\delta t) - \underset{\rightarrow}{\boldsymbol{p}}(t)}{\delta t} = \sigma_a v^2 \cos(\alpha_a) dS \hat{\underset{\rightarrow}{\boldsymbol{v}}} = -d\underset{\rightarrow}{\boldsymbol{f}}_{aer} \tag{16.8}$$

Integrating this result over the entire "wetted" surface, parts of the surface area that are actually in contact with the streamtube considered, provides the total aerodynamic force on the spacecraft:

$$\vec{\boldsymbol{f}}_{aer} = -\sigma_a v^2 \hat{\vec{\boldsymbol{v}}} \iint\limits_{S_{wet}} \hat{\vec{\boldsymbol{v}}} \cdot \hat{\vec{\boldsymbol{n}}}\; dS \tag{16.9}$$

where S_{wet} includes only portions of the surface facing the flow, hence satisfying $\hat{\vec{\boldsymbol{v}}} \cdot \hat{\vec{\boldsymbol{n}}} \geqslant 0$. Finally, the total aerodynamic torque (about the centre of mass) is determined:

$$\vec{\boldsymbol{\tau}}_{aer} = \vec{\boldsymbol{c}}_{p_{aer}} \times \vec{\boldsymbol{f}}_{aer} \quad , \quad \vec{\boldsymbol{c}}_{p_{aer}} \triangleq \frac{\iint\limits_{S_{wet}} (\hat{\vec{\boldsymbol{v}}} \cdot \hat{\vec{\boldsymbol{n}}}) \vec{\boldsymbol{\rho}}\; dS}{\iint\limits_{S_{wet}} \hat{\vec{\boldsymbol{v}}} \cdot \hat{\vec{\boldsymbol{n}}}\; dS} \tag{16.10}$$

where $\vec{\boldsymbol{\rho}}$ and $\vec{\boldsymbol{c}}_{p_{aer}}$ are the position vectors of a differential surface element and the centre of aerodynamic pressure, measured from the centre of mass. To extend this result to spinning spacecraft, the change in the air molecules' relative velocity from one point on the spacecraft to the other must be considered: $\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}_{a_{\otimes}} - \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{\rho}}$, where $\vec{\boldsymbol{v}}_{a_{\otimes}}$ is the atmospheric velocity with respect to the centre of mass and $\vec{\boldsymbol{\omega}}$ is the spacecraft's angular velocity with respect to the atmosphere, but it could be approximated as its inertial angular velocity.

Note: These relationships are not valid for launch or landing. Regular aerodynamics (with continuum fluid mechanics) relationships, similar to those in LAUNCH VEHICLE DYNAMICS, should be used for those stages of the mission.

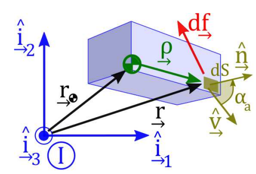

Figure 16.5: atmospheric drag

Solar Radiation Pressure Torque

Transfer of momentum from photons impending on the surface of spacecraft causes small forces that, depending on the size and reflective properties of the surface, can add up to become noticeable. Solar radiation forces fall off proportionally to $1/r^2$, where r is the distance from Sun. In order to model solar radiation torques in this section, total absorption of the light rays by the surface material is assumed, even though there will always be at least some reflection involved. The differential force is given by:

$$d\vec{\boldsymbol{f}}_{sol} = -P_s \hat{\vec{\boldsymbol{s}}}\; dA = -P_s (\hat{\vec{\boldsymbol{s}}} \cdot \hat{\vec{\boldsymbol{n}}}) \hat{\vec{\boldsymbol{s}}}\; dS \quad , \quad \cos(\alpha_s) = \hat{\vec{\boldsymbol{s}}} \cdot \hat{\vec{\boldsymbol{n}}} \tag{16.11}$$

where, once again and as depicted in Figure 16.6, dA represents the projection of dS, this time in the Sun's direction, represented by $\hat{\vec{\boldsymbol{s}}}$. Solar radiation pressure, P_s, is a function of radial distance from Sun, and $P_s \approx 4.5 \times 10^{-6}\ \mathrm{N/m^2}$ close to Earth's orbit, varying by approximately 6% over the course of Earth's elliptic orbit. Integrating Eq. (16.11) over the entire "wetted" surface, parts of the surface area that are lit by sunlight, provides the total solar radiation force on the spacecraft:

$$\vec{\boldsymbol{f}}_{sol} = -P_s \hat{\vec{\boldsymbol{s}}} \iint\limits_{S_{lit}} \hat{\vec{\boldsymbol{s}}} \cdot \hat{\vec{\boldsymbol{n}}}\; dS \tag{16.12}$$

where S_{lit} includes only portions of the surface facing Sun, hence satisfying $\hat{\vec{\boldsymbol{s}}} \cdot \hat{\vec{\boldsymbol{n}}} \geqslant 0$. Finally, the total solar radiation torque (about the centre of mass) is determined:

$$\vec{\boldsymbol{\tau}}_{sol} = \vec{\boldsymbol{c}}_{p_{sol}} \times \vec{\boldsymbol{f}}_{sol} \quad , \quad \vec{\boldsymbol{c}}_{p_{sol}} \triangleq \frac{\iint_{S_{lit}} (\hat{\vec{\boldsymbol{s}}} \cdot \hat{\vec{\boldsymbol{n}}}) \vec{\boldsymbol{\rho}} \, dS}{\iint_{S_{lit}} \hat{\vec{\boldsymbol{s}}} \cdot \hat{\vec{\boldsymbol{n}}} \, dS} \tag{16.13}$$

where $\vec{\boldsymbol{\rho}}$ and $\vec{\boldsymbol{c}}_{p_{sol}}$ are the position vectors of a differential surface element and the centre of solar radiation pressure, measured from the centre of mass.

Note: In addition to the solar effects, Earth-orbiting spacecraft are also affected by both reflection from and emission of radiation by Earth.

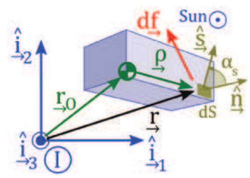

Figure 16.6: Radiation

Gravity Gradient Torque

Resulting from non-uniform spatial distribution of the gravitational field and the varying magnitude and direction of the gravitational forces on spacecraft's body, gravity gradient exerts disturbance torques that modify the spacecraft's rotational motion.

The differential gravitational force on an infinitesimal mass element, dm, located at a relative position of $\vec{\boldsymbol{\rho}}$ from the spacecraft's centre of mass as shown in Figure (16.7), is given by:

$$d\vec{\boldsymbol{f}} = \frac{-GM \, dm}{r^3} \vec{\boldsymbol{r}} \quad , \quad \vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_{\oplus} + \vec{\boldsymbol{\rho}} \tag{16.14}$$

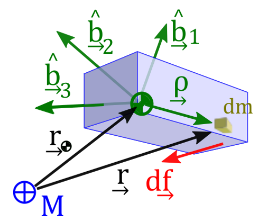

Figure 16.7: Gravity

where $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{r}}_{\oplus}$ are the position, relative to the centre of Earth, of the mass element and the spacecraft's centre of mass. The squared norm of $\vec{\boldsymbol{r}}$ can be expressed as:

$$r^2 = \vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{r}} = (\vec{\boldsymbol{r}}_{\oplus} + \vec{\boldsymbol{\rho}}) \cdot (\vec{\boldsymbol{r}}_{\oplus} + \vec{\boldsymbol{\rho}}) = r_{\oplus}^2 + 2\vec{\boldsymbol{\rho}} \cdot \vec{\boldsymbol{r}}_{\oplus} + \rho^2 \tag{16.15}$$

taking which to the power of $(-3/2)$ yields:

$$r^{-3} = (r_{\oplus}^2 + 2\vec{\boldsymbol{\rho}} \cdot \vec{\boldsymbol{r}}_{\oplus} + \rho^2)^{-3/2} = r_{\oplus}^{-3} \left(1 + \frac{2\vec{\boldsymbol{\rho}} \cdot \vec{\boldsymbol{r}}_{\oplus}}{r_{\oplus}^2} + \frac{\rho^2}{r_{\oplus}^2}\right)^{-3/2} \tag{16.16}$$

but noting that $\rho \ll r_{\oplus}$, we can make use of the approximation $(1+x)^n \approx 1 + nx$ for $|x| \ll 1$ to expand Eq. (16.16) as follows:

$$r^{-3} \approx r_{\oplus}^{-3} \Big[1 - \frac{3}{2}\Big(\frac{2\vec{\boldsymbol{\rho}} \cdot \vec{\boldsymbol{r}}_{\oplus}}{r_{\oplus}^2} + \frac{\rho^2}{r_{\oplus}^2}\Big)\Big] \approx r_{\oplus}^{-3} \Big[1 - 3\frac{\vec{\boldsymbol{\rho}} \cdot \vec{\boldsymbol{r}}_{\oplus}}{r_{\oplus}^2} + \underbrace{O\Big(\frac{\rho^2}{r_{\oplus}^2}\Big)}_{\approx 0}\Big] \tag{16.17}$$

substituting which back into Eq. (16.14) and letting $\mu \triangleq GM$ yields:

$$d\vec{\boldsymbol{f}} \approx \frac{-\mu \, dm}{r_{\oplus}^3} \Big(1 - 3\frac{\vec{\boldsymbol{\rho}} \cdot \vec{\boldsymbol{r}}_{\oplus}}{r_{\oplus}^2}\Big)(\vec{\boldsymbol{r}}_{\oplus} + \vec{\boldsymbol{\rho}}) \tag{16.18}$$

Lastly, the total gravitational torque on the spacecraft about its centre of mass can be computed:

$$\underset{\rightarrow}{\boldsymbol{\tau}}_{gg} = \iiint\limits_V \underset{\rightarrow}{\boldsymbol{\rho}} \times d\underset{\rightarrow}{\boldsymbol{f}} = \frac{-\mu}{r_{\oplus}^3} \iiint\limits_V \left(1 - 3\frac{\underset{\rightarrow}{\boldsymbol{\rho}} \cdot \underset{\rightarrow}{\boldsymbol{r}}_{\oplus}}{r_{\oplus}^2}\right)(\cancelto{\underset{\rightarrow}{\mathbf{0}}}{\underset{\rightarrow}{\boldsymbol{\rho}} \times \underset{\rightarrow}{\boldsymbol{\rho}}} + \underset{\rightarrow}{\boldsymbol{\rho}} \times \underset{\rightarrow}{\boldsymbol{r}}_{\oplus})\sigma(\underset{\rightarrow}{\boldsymbol{\rho}})\, dV \tag{16.19}$$

expanding and simplifying which, noting that $\underset{\rightarrow}{\boldsymbol{r}}_{\oplus}$ remains constant over the volume, results in:

$$\underset{\rightarrow}{\boldsymbol{\tau}}_{gg} = \frac{-\mu}{r_{\oplus}^3} \cancelto{\underset{\rightarrow}{\mathbf{0}}}{\iiint\limits_V \underset{\rightarrow}{\boldsymbol{\rho}}\sigma(\underset{\rightarrow}{\boldsymbol{\rho}})\, dV} \times \underset{\rightarrow}{\boldsymbol{r}}_{\oplus} + \frac{3\mu}{r_{\oplus}^5} \iiint\limits_V (\underset{\rightarrow}{\boldsymbol{\rho}} \times \underset{\rightarrow}{\boldsymbol{r}}_{\oplus})(\underset{\rightarrow}{\boldsymbol{\rho}} \cdot \underset{\rightarrow}{\boldsymbol{r}}_{\oplus})\sigma(\underset{\rightarrow}{\boldsymbol{\rho}})\, dV \tag{16.20}$$

where the first term vanishes by definition of centre of mass from DYNAMICS and our choice of $O \equiv \oplus$. Upon resolving all vectors in a body-fixed frame, $\mathscr{F}_B$, with its origin at O, Eq. (16.20) takes on the following referential form after using $\boldsymbol{\rho}^{\times}\boldsymbol{r}_{\oplus} = -\boldsymbol{r}_{\oplus}^{\times}\boldsymbol{\rho}$:

$$\boldsymbol{\tau}_{gg} = \frac{-3\mu}{r_{\oplus}^5}\boldsymbol{r}_{\oplus}^{\times} \iiint\limits_V \boldsymbol{\rho}\boldsymbol{\rho}^{\mathsf{T}}\sigma(\boldsymbol{\rho})\, dV \boldsymbol{r}_{\oplus} = \frac{-3\mu}{r_{\oplus}^5}\boldsymbol{r}_{\oplus}^{\times} \cancelto{-\boldsymbol{J}}{\iiint\limits_V \boldsymbol{\rho}^{\times}\boldsymbol{\rho}^{\times}\sigma(\boldsymbol{\rho})\, dV} \boldsymbol{r}_{\oplus} - \frac{3\mu}{r_{\oplus}^5} \iiint\limits_V \rho^2 \cancelto{\mathbf{0}}{\boldsymbol{r}_{\oplus}^{\times}\boldsymbol{r}_{\oplus}}\sigma(\underset{\rightarrow}{\boldsymbol{\rho}})\, dV \tag{16.21}$$

where the definition of the moment of inertia matrix from DYNAMICS, and the identity $\boldsymbol{b}\boldsymbol{a}^{\mathsf{T}} = \boldsymbol{a}^{\times}\boldsymbol{b}^{\times} + (\boldsymbol{a}^{\mathsf{T}}\boldsymbol{b})\mathbf{1}$ from FUNDAMENTALS are used, setting $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{\rho}$ for the latter. The gravity gradient torque is, therefore, represented in the following form:

$$\boldsymbol{\tau}_{gg} = \frac{3\mu}{r_{\oplus}^5}\boldsymbol{r}_{\oplus}^{\times}\boldsymbol{J}\boldsymbol{r}_{\oplus} \tag{16.22}$$

where all column matrices are their corresponding vectors' expressions in $\mathscr{F}_B$.

References

[Hughes] Hughes, P. C., *Spacecraft Attitude Dynamics*, Dover Publications Inc., New York, Chap. 8.