Lecture 18

Active Attitude Control

FUNDAMENTALS of active attitude control are discussed. Mathematical basics of control systems, including Laplace transform and its properties, transfer functions, and interconnections of systems are reviewed. Lastly, simple feedback attitude control laws are presented, and the step input response of a proportional-derivative controlled spacecraft is examined.

Overview

Passive control schemes, such as using gravity gradient stabilization, can only provide coarse attitude control that may not meet some missions' pointing or rotational rate requirements. *Active* control is, therefore, required to continuously measure and correct for, using judiciously modified forces and torques, deviations from the desired attitude.

In general, the objective of active attitude control is to eliminate or minimize the error, $\delta\boldsymbol{\theta}(t) = \boldsymbol{\theta}(t) - \boldsymbol{\theta}_{ref}(t)$, taking into account that measurements of the output, $\tilde{\boldsymbol{\theta}}$, inevitably include sensor noise, $\delta\boldsymbol{\theta}_n$. The reference attitude, represented by $\boldsymbol{\theta}_{ref}$, is either pre-programmed or commanded on-line by mission crew or ground operators. To achieve this objective, control torques, $\boldsymbol{\tau}_c(t)$, are applied via actuators, in conjunction with external disturbance torques, $\boldsymbol{\tau}_d(t)$, some sources of which were discussed in DISTURBANCE TORQUES. An overview of the attitude control system of interest is provided in Figure 18.1.

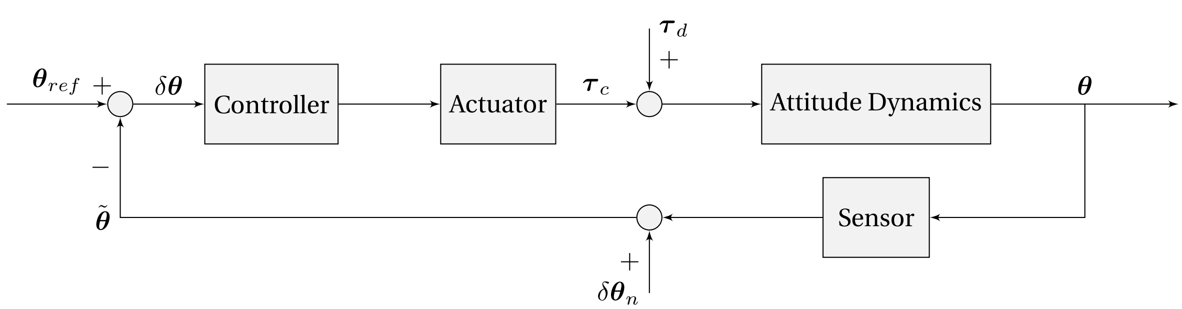


Figure 18.1: Overview of Spacecraft Attitude Control System

Laplace Transform

Recall Laplace transform that maps a function from t-domain to s-domain:

$$\mathscr{L}[f(t)] = \int_0^\infty f(t)e^{-st}\, dt \triangleq \mathbb{f}(s) \tag{18.1}$$

which can be used to transform linear ODEs to algebraic equations using its differentiation properties. For the function's first time-derivative, for example, we have:

$$\mathscr{L}[\dot{f}(t)] = \int_0^\infty \dot{f}(t)e^{-st}\, dt = f(t)e^{-st}\Big|_0^\infty - \int_0^\infty f(t)\big(-se^{-st}\, dt\big) = -f(0) + s\mathbb{f}(s) \tag{18.2}$$

where integration by parts is used. Successive applications of this property yields, for the n^{th} derivative:

$$\mathscr{L}[f^{(n)}(t)] = s^n\mathbb{f}(s) - \sum_{k=1}^{n} s^{k-1}f^{(n-k)}(0) \tag{18.3}$$

Laplace transform enjoys a number of other important properties, such as:

$$\mathscr{L}[af(t) + bg(t)] = a\mathbb{f}(s) + b\mathbb{g}(s) \tag{18.4a}$$

$$\mathscr{L}[f(t) * g(t)] = \mathscr{L}\Big[\int_0^t f(\tau)g(t-\tau)\, d\tau\Big] = \mathbb{f}(s)\mathbb{g}(s) \tag{18.4b}$$

$$f(0^+) = \lim_{s\to\infty} s\mathbb{f}(s) \ , \quad \lim_{t\to\infty} f(t) = \lim_{s\to 0} s\mathbb{f}(s) \tag{18.4c}$$

which are the linearity property, convolution theorem, and initial and final value theorems, respectively.

Transfer Function

Any control system can be represented using an operator that maps the input(s), $u(t)$, to the output(s), $y(t)$. For the control laws of interest to us, this mapping can be represented in the s-domain using a transfer function (or matrix, for multiple inputs and outputs):

$$\mathbb{y}(s) = \mathbb{G}(s)\mathbb{u}(s) \tag{18.5}$$

where, in general, the transfer function, $\mathbb{G}(s)$, is of the following form:

$$\mathbb{G}(s) = \frac{a_0s^m + a_1s^{m-1} + \cdots + a_m}{s^n + b_1s^{n-1} + \cdots + b_n} \tag{18.6}$$

the zeros of the numerator of which provide the system's *zeros*, and those of the denominator are called the system's *poles*. For stability, no poles should be on the right half of the complex plane (positive real parts), and no repeated poles should appear on the imaginary axis. The input response of the system in the t-domain can be obtained by $y(t) = \mathscr{L}^{-1}[\mathbb{G}\mathbb{u}]$, where, for example, $\mathbb{u} = 1$ for an impulsive input, $\mathbb{u} = 1/s$ for a step input, and $\mathbb{u} = 1/s^2$ for a ramp input.

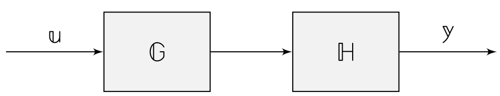
Interconnection of Systems

The overall transfer functions associated with various interconnections of multiple control systems, represented in Figure (18.3) using block diagrams, are provided by:

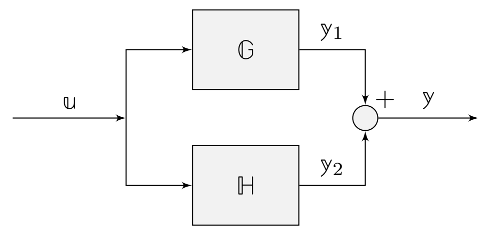
(a) Series Connection: $\mathbb{y} = (\mathbb{G}\mathbb{H})\mathbb{u}$

(b) Parallel Connection: $\mathbb{y} = (\mathbb{G} + \mathbb{H})\mathbb{u}$

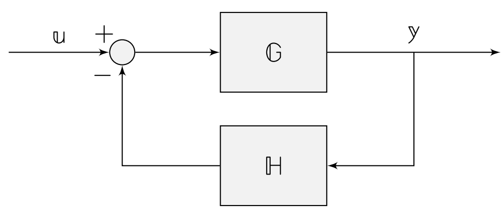
(c) Feedback Interconnection: $\mathbb{y} = \left(\dfrac{\mathbb{G}}{1 + \mathbb{G}\mathbb{H}}\right)\mathbb{u}$



(a) Series Connection



(b) Parallel Connection



(c) Feeback Interconnection

Figure 18.3: Interconnection of Two Systems

In preparation for the active attitude control problem, we can treat the reference desired output as an input to the system, and let the input of the controller be the error (the difference between the desired and the actual system output): defining $\delta y(t) = y_{ref} - y(t)$ with Laplace transform $\delta\mathbb{y}(s) = \mathbb{y}_{ref}(s) - \mathbb{y}(s)$, and letting the controller react to the error as $\mathbb{u}_c = \mathbb{K}\delta\mathbb{y}$, we have the feedback system shown below, where $\mathbb{H}(s) = 1$ for this case and the process noise, $\mathbb{u}_d(s)$, is also included.

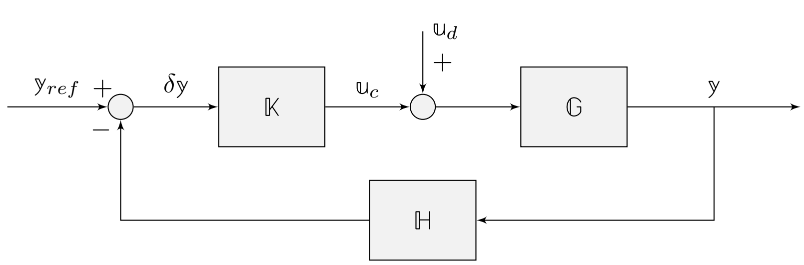


Figure 18.4: Feedback System using Reference Signal and Output Error

Attitude Control Problem

We consider the problem of 3-axis attitude control of a principal axes frame, $\mathscr{F}_P$, with respect to an inertially-fixed nominal frame, $\mathscr{F}_I$. Euler's rigid body equations form DYNAMICS are recalled:

$$\boldsymbol{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}^{\times}\boldsymbol{I}\boldsymbol{\omega} = \boldsymbol{\tau}_c + \boldsymbol{\tau}_d \tag{18.7}$$

where the subscripts 'c' and 'd' distinguish the control torques from the disturbance torques. Assuming infinitesimally small angles, $\boldsymbol{\omega} \approx \delta\dot{\boldsymbol{\theta}}$ from KINEMATICS, which yields the following linearized form of the equations of motion in Eq. (18.7):

$$\boldsymbol{I}\delta\ddot{\boldsymbol{\theta}} + \underbrace{\delta\dot{\boldsymbol{\theta}}^{\times}\boldsymbol{I}\delta\dot{\boldsymbol{\theta}}}_{\approx \mathbf{0}} \approx \boldsymbol{I}\delta\ddot{\boldsymbol{\theta}} \approx \boldsymbol{\tau}_c + \boldsymbol{\tau}_d \tag{18.8}$$

where small rates are also assumed. Since these equations are now decoupled, they can be considered separately. The following scalar equation for each axis will, therefore, be the focus of the rest of this lesson:

$$I\ddot{\theta}(t) \approx \tau_c(t) + \tau_d(t) \tag{18.9}$$

where $\delta\theta$ is replaced by θ for brevity. Taking the Laplace transform of Eq. (18.9) yields:

$$I\ddot{\theta}(t) = \tau_c(t) + \tau_d(t) \quad \overset{\mathscr{L}}{\Longrightarrow} \quad I\big[s^2\theta(s) - s\theta(0) - \dot{\theta}(0)\big] = \tau_c(s) + \tau_d(s) \tag{18.10}$$

Feedback Control

Since we expect non-zero initial conditions (ICs) to eventually disappear and we are more interested in the input response, we let $\theta(0) = \dot{\theta}(0) = 0$, which simplifies Eq. (18.10) as follows:

$$\theta = \frac{\tau_c + \overset{\approx 0}{\tau_d}}{s^2 I} \approx \frac{1}{Is^2}\tau_c \quad \Rightarrow \theta = \mathbb{G}\tau_c \ , \quad \mathbb{G} \triangleq \frac{1}{Is^2} \tag{18.11}$$

where small disturbance torques, at least compared to the control inputs, are assumed. The plant transfer function is represented by $\mathbb{G}$, which has a pair of repeated poles at $s = 0$, hence making such an open-loop system unstable. To remedy this situation, we incorporate feedback control.

P Control

The proportional control torque is set to be proportional to the attitude error. Disregarding disturbances and using zero ICs, we have:

$$I\ddot{\theta} = \tau_c = K_p\big(\theta_{ref} - \theta\big) \quad \overset{\mathscr{L}}{\Longrightarrow} \quad Is^2\theta = K_p\big(\theta_{ref} - \theta\big) \tag{18.12}$$

the controller and the closed-loop transfer functions of which are found to be as follow:

$$\tau_c = \mathbb{K}e_c \ , \ e_c \triangleq \theta_{ref} - \theta \ , \ \mathbb{K} \triangleq K_p \ \text{ and } \ \theta = \frac{K_p}{Is^2 + K_p}\theta_{ref} = \frac{\mathbb{G}\mathbb{K}}{1 + \mathbb{G}\mathbb{K}}\theta_{ref} \ , \ \mathbb{G} \triangleq \frac{1}{Is^2} \tag{18.13}$$

where the same plant transfer function $\mathbb{G}$ as in Eq. (18.11) is used. The closed-loop control law can also be visualized, as in Figure 18.5, using a negative-feedback diagram involving a series interconnection of $\mathbb{G}$ and $\mathbb{K}$, and a unity feedback gain of $\mathbb{H} = 1$. The combined transfer function $\mathbb{GK}$ is known as the open-loop transfer function, the poles of which can also be used for stability analysis purposes.

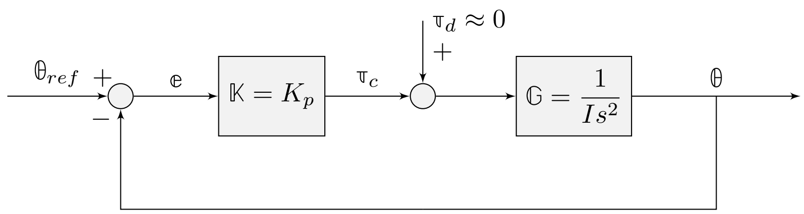


Figure 18.5: Feedback System using Reference Signal and Output Error

PD Control

The proportional-derivative control torque is set to receive proportional contributions from both the attitude error and the attitude rate error. Disregarding disturbances and using zero ICs, we have:

$$I\ddot{\theta} = \tau_c = K_p\big(\theta_{ref} - \theta\big) + K_d\big(\dot{\theta}_{ref} - \dot{\theta}\big) \;\overset{\mathscr{L}}{\Longrightarrow}\; Is^2\theta(s) = K_p\big(\theta_{ref} - \theta\big) + K_d s\big(\theta_{ref} - \theta\big) \tag{18.14}$$

the controller and the closed-loop transfer functions of which are found to be as follow:

$$\tau_c = \mathbb{K}e_c \;,\; e_c \triangleq \theta_{ref} - \theta \;,\; \mathbb{K} \triangleq K_d s + K_p \;\text{ and }\; \theta = \frac{K_d s + K_p}{Is^2 + K_d s + K_p}\theta_{ref} = \frac{\mathbb{GK}}{1 + \mathbb{GK}}\theta_{ref} \;,\; \mathbb{G} \triangleq \frac{1}{Is^2} \tag{18.15}$$

where the same plant transfer function $\mathbb{G}$ as in Eq. (18.11) is used. Similar comments to the P law can be made about the negative-feedback interconnections represented by this relationship.

We now reconsider the disturbance torque, $\tau_d(t) \neq 0$, and assume a constant reference input, θ_{ref}, that implies $\dot{\theta}_{ref} = 0$. We furthermore consider the steady-state case with $\dot{\theta} = \ddot{\theta} = 0$ as $t \to \infty$, using which Eq. (18.14) becomes:

$$I\cancelto{0}{\ddot{\theta}(t)} = \tau_c(t) + \tau_d(t) = K_p\big(\theta_{ref} - \theta\big) - K_d\cancelto{0}{\dot{\theta}} + \tau_d \;\Rightarrow\; e \triangleq \theta_{ref} - \theta = \frac{-\tau_d}{K_p} \tag{18.16}$$

This implies that although PD law eliminates the oscillations in attitude in the long term, it cannot remove the steady-state errors. This provides the motivation behind adding an integral term.

Response to Step Attitude Command

Let us focus on one of the most common attitude control laws, namely PD control. From Eq. (18.15), the corresponding closed-loop transfer function is given by:

$$\frac{\theta(s)}{\theta_{ref}(s)} = \frac{K_d s + K_P}{Is^2 + K_d s + K_p} = \frac{\frac{K_d}{I}s + \frac{K_P}{I}}{s^2 + \frac{K_d}{I}s + \frac{K_P}{I}} \tag{18.17}$$

which can be rewritten in the following form:

$$\frac{\Theta(s)}{\Theta_{ref}(s)} = \frac{2\zeta\omega_0 s + \omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \;\; , \;\; \zeta \triangleq \frac{K_d}{2}\sqrt{\frac{1}{K_P I}} \;\; , \;\; \omega_0 = \sqrt{\frac{K_p}{I}} \tag{18.18}$$

where ζ is known as the system's *damping ratio* and ω_0 is its *undamped natural frequency*. The characteristic equation of the system and its solutions, known as the closed-loop poles, are given by:

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0 \;\; \Rightarrow \;\; s_{1,2} = \frac{-2\zeta\omega_0 \pm \sqrt{4\zeta^2\omega_0^2 - 4\omega_0^2}}{2} = -\zeta\omega_0 \mp i\omega_d \;\; , \;\; \omega_d \triangleq \omega_0\sqrt{1-\zeta^2} \tag{18.19}$$

where ω_d is the system's *damped natural frequency*. The size of the damping ratio dictates the system's overall behaviour:

- $0 = \zeta$: purely imaginary pair of poles resulting in pure oscillation (*undamped*)
- $0 < \zeta < 1$: complex conjugate pair of poles resulting in decaying oscillation (*underdamped*)
- $1 = \zeta$: purely real (negative) poles resulting in boundary of exponential decay (*critically damped*)
- $1 < \zeta$: distinct purely real (negative) poles resulting in exponential decay (*overdamped*)

We now consider, as an input, a step attitude command of $\theta_{ref}(t) = CH(t) = C$ for $t \geqslant 0$, where $H(t)$ is the Heaviside function and C is the constant step amplitude. Recall that the Laplace transform of this input is $\Theta_{ref}(s) = C/s$.

Transient Response

The closed-loop transfer function in Eq. (18.18) yields, for the Laplace transform of the normalized output:

$$\frac{\Theta}{C} = \frac{2\zeta\omega_0 s + \omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \cdot \frac{1}{s} = \frac{1}{s} + \frac{-s}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \tag{18.20}$$

where partial fractions are used to separate the rational functions into additive terms. Further manipulation of Eq. (18.20), by completing the square in the denominator and rewriting the numerator to reach familiar Laplace transform functions, yields:

$$\frac{\Theta}{C} = \frac{1}{s} - \frac{s + \zeta\omega_0}{(s + \zeta\omega_0)^2 + \omega_d^2} + \frac{\zeta\omega_0}{\omega_d} \cdot \frac{\omega_d}{(s + \zeta\omega_0)^2 + \omega_d^2} \tag{18.21}$$

where $\omega_d^2 = \omega_0^2(1-\zeta^2)$ from Eq. (18.19) is used. Taking the inverse Laplace transform of Eq. (18.21) produces the time-domain response of the system to a step function for $t \geqslant 0$:

$$\theta(t) = C - Ce^{-\zeta\omega_0 t}\Big(\cos(\omega_d t) - \frac{\zeta\omega_0}{\omega_d}\sin(\omega_d t)\Big) \tag{18.22}$$

where the property that $\mathscr{L}[e^{-at}f(t)] = \mathbb{F}(s+a)$ is used. It is evident that the desired behaviour can be achieved by setting ω_0 and ζ according to the mission requirements.

The following specifications pertinent to the transient response of such systems are commonly used in literature:

- *rise time* to reach the final value, $\theta(t_R) = C$, or to go from 10% to 90% of the step input

- *settling time* to reach and remain within 2% of the final value
- *overshoot* reported as maximum percentage of output exceeding the final value

Note: Refer to Section 17.7.1 of *Spacecraft Dynamics and Control: an Introduction* for details on how each of these parameters can be estimated.

Steady-State Response

The final value theorem of Laplace transform from Eq. (18.4c) can be used to study the steady-state value of the output subject to the PD law of interest:

$$\lim_{t\to\infty}\theta(t) = \lim_{s\to 0} s\theta(s) = \lim_{s\to 0} \cancel{s}\frac{2\zeta\omega_0 s + \omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}\cdot\frac{C}{\cancel{s}} = \frac{C\cancel{\omega_0^2}}{\cancel{\omega_0^2}} = C \tag{18.23}$$

which establishes the fact that, in the absence of disturbance torques, a PD law would suffice for the spacecraft to asymptotically approach a commanded attitude of $\theta_{ref} = C$.