Lecture 19

Bias-Momentum Stabilization

Dynamics of spacecraft under gravity gradient with actively-controlled wheels are considered. Linearized pitch and roll/yaw equations of motion are used in conjunction with modified P- and PD-type feedback control laws, and the system's resulting stability properties and steady-state behaviour are studied.

Overview

Bias-momentum-stabilized spacecraft are similar to gyrostats considered in Dual-Spin Stabilization, but instead of large external rotors, they have relatively small rapidly spinning internal wheels that provide gyricity beneficial to attitude stabilization. They also tend to rely more heavily on active control than gyrostats typically do, and bias-momentum stabilization can generally be considered as an amalgam of passive dual-spin stabilization and active attitude control in the presence of gravity gradient.

The following advantages motivate bias-momentum stabilization:

- providing short term stability against disturbances, similarly to spin stabilization
- increasing roll/yaw coupling, hence reducing the need for yaw sensing, which is difficult because the body-fixed frame measurements of the spacecraft's position vector relative to Earth do not depend on yaw
- enhancing gravity gradient stabilization as a consequence of having a wheel nominally aligned with the orbiting frame's 2-axis (pitch)

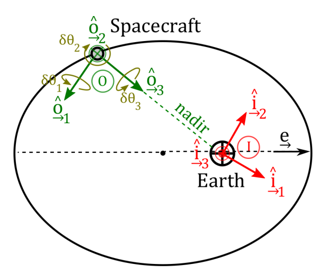

Figure 19.1: Orbiting (O) and Inertial (I) Frames

Note: As mentioned before, a bias-momentum wheel that remains aligned with the pitch axis and only changes rate is known as a "reaction wheel", while one that changes direction from roll/yaw to pitch is a "control moment gyro" (CMG).

Analogously to Gravity Gradient Stabilization, The following reference frames, shown in Figure 19.1, are used for the purpose of this study:

- $\mathscr{F}_I$: inertial frame fixed to (but not rotating with) Earth

- $\mathscr{F}_O$: orbiting frame, with origin fixed to spacecraft, 3-axis towards Earth's centre, 2-axis anti-parallel to orbital angular momentum, $\underrightarrow{\boldsymbol{h}}$

- $\mathscr{F}_B$: body-fixed frame, with origin at spacecraft's centre of mass

A circular orbit is assumed, with a mean motion of $\omega_0 = \sqrt{\mu/r_\oplus^3}$. In addition, the wheel's spin axis is taken to be along the negative pitch axis, $\hat{\boldsymbol{a}} = -\hat{\boldsymbol{o}}_2$, and its angular momentum is given by $\boldsymbol{h}_s = I_s\omega_s\hat{\boldsymbol{a}}$. The wheel is assumed to be spinning rapidly enough to justify taking $h_s \gg I_i\omega_0,\ i \in \{1,2,3\}$.

Based on FUNDAMENTALS, the spacecraft's attitude with respect to the nominal orbiting frame can be described using a 3-2-1 rotation matrix:

$$\boldsymbol{C}_{BO} = \boldsymbol{C}_1(\delta\theta_1)\boldsymbol{C}_2(\delta\theta_2)\boldsymbol{C}_3(\delta\theta_3) \quad \Rightarrow \quad \boldsymbol{C}_{BO} \approx \mathbf{1} - \delta\boldsymbol{\theta}^\times \approx \begin{bmatrix} 1 & \delta\theta_3 & -\delta\theta_2 \\ -\delta\theta_3 & 1 & \delta\theta_1 \\ \delta\theta_2 & -\delta\theta_1 & 1 \end{bmatrix} \quad , \quad \delta\boldsymbol{\theta} \triangleq \begin{bmatrix} \delta\theta_1 \\ \delta\theta_2 \\ \delta\theta_3 \end{bmatrix} \tag{19.1}$$

where $\delta\theta_1$, $\delta\theta_2$, and $\delta\theta_3$ are the infinitesimal roll, pitch, and yaw angles, respectively. Small Euler angle approximations are used in Eq. (19.1).

Equations of Motion

We recall the following results (all expressed in $\mathscr{F}_B$) from GRAVITY GRADIENT STABILIZATION, with $\delta\boldsymbol{\theta}$ representing the small roll, pitch, and yaw angles:

$$\boldsymbol{\omega}^{BI} = \cancelto{\approx\delta\dot{\boldsymbol{\theta}}}{\boldsymbol{\omega}^{BO}} + \boldsymbol{C}_{BO}\begin{bmatrix} 0 \\ -\omega_0 \\ 0 \end{bmatrix} = \begin{bmatrix} \delta\dot{\theta}_1 - \omega_0\delta\theta_3 \\ \delta\dot{\theta}_2 - \omega_0 \\ \delta\dot{\theta}_3 + \omega_0\delta\theta_1 \end{bmatrix} \quad , \quad \boldsymbol{\tau}_{gg} = 3\omega_0^2\begin{bmatrix} (I_3 - I_2)\delta\theta_1 \\ (I_3 - I_1)\delta\theta_2 \\ (I_1 - I_2)\cancelto{\approx 0}{\delta\theta_1\delta\theta_2} \end{bmatrix} \tag{19.2}$$

Euler's equations of motion in the presence of gravity gradient, control, and disturbance torques (namely $\boldsymbol{\tau}_{gg}$, $\boldsymbol{\tau}_c$, and $\boldsymbol{\tau}_d$, respectively) are given by:

$$\dot{\boldsymbol{h}} + \boldsymbol{\omega}^\times\boldsymbol{h} = \boldsymbol{\tau}_{gg} + \boldsymbol{\tau}_c + \boldsymbol{\tau}_d \ ; \ \ \boldsymbol{h} = \boldsymbol{I}\boldsymbol{\omega}^{BI} + \boldsymbol{h}_s \ , \ \ \boldsymbol{h}_s = \begin{bmatrix} 0 \\ -I_s\omega_s \\ 0 \end{bmatrix} \tag{19.3}$$

where, for now, no active torque about the pitch axis is assumed; that is, $\tau_{c_2} = 0$.

Expanding and rearranging Eq. (19.3) yields the equations of motion:

$$I_1\delta\ddot{\theta}_1 - \big[(I_1 - I_2 + I_3)\omega_0 - h_s\big]\delta\dot{\theta}_3 + \big[4\omega_0^2(I_2 - I_3) + h_s\omega_0\big]\delta\theta_1 = \tau_{c_1} + \tau_{d_1} \tag{19.4a}$$

$$I_2\delta\ddot{\theta}_2 + 3\omega_0^2(I_1 - I_3)\delta\theta_2 = \dot{h}_s + \tau_{d_2} \tag{19.4b}$$

$$I_3\delta\ddot{\theta}_3 + \big[(I_1 - I_2 + I_3)\omega_0 - h_s\big]\delta\dot{\theta}_1 + \big[\ \omega_0^2(I_2 - I_1) + h_s\omega_0\big]\delta\theta_3 = \tau_{c_3} + \tau_{d_3} \tag{19.4c}$$

which resemble the equations of motion involved in gravity gradient stabilization, but with the h_s effects. The $\dot{h}_s$ term behaves as pitch control provided by gyric effects of the wheel spinning in the pitch direction.

Assuming a rapidly spinning wheel, we let $h_s = I_s\omega_s \gg I_i\omega_0$ for $i \in \{1,2,3\}$, which simplifies the equations of motion in Eq. (19.4) for the h_s terms inside the brackets dominate:

$$I_1\delta\ddot{\theta}_1 + h_s\delta\dot{\theta}_3 + h_s\omega_0\delta\theta_1 = \tau_{c_1} + \tau_{d_1} \tag{19.5a}$$

$$I_2\delta\ddot{\theta}_2 + 3\omega_0^2(I_1 - I_3)\delta\theta_2 = \dot{h}_s + \tau_{d_2} \tag{19.5b}$$

$$I_3\delta\ddot{\theta}_3 - h_s\delta\dot{\theta}_1 + h_s\omega_0\delta\theta_3 = \tau_{c_3} + \tau_{d_3} \tag{19.5c}$$

which has its pitch equation uncoupled, and its roll/yaw equations coupled together. Similarly to GRAVITY GRADIENT STABILIZATION, we now treat the control about pitch and roll/yaw axes separately.

Pitch Control

Let us treat $\dot{h}_s$ as pitch control torque, $\tau_{c_2} \triangleq \dot{h}_s$, which can be modified by the wheel's rotation. For simplicity of notation, we replace $\delta\theta_2$ with θ_2, and considering a stabilization problem in the presence of disturbances, we let $\theta_{2_{ref}} = 0$ and $\theta_2(0) = \dot{\theta}_2(0) = 0$. Taking the Laplace transform of the motion equation in Eq. (19.5b) results in:

$$(I_2 s^2 + C)\Theta_2 = \mathbb{T}_{c_2} + \mathbb{T}_{d_2} \quad , \quad C \triangleq 3\omega_0^2(I_1 - I_3) \tag{19.6}$$

Consider the following modified PD control law and its Laplace transform:

$$\tau_{c_2} = K_p(\cancelto{0}{\theta_{2_{ref}}} - \theta_2) + K_d(\cancelto{0}{\dot{\theta}_{2_{ref}}} - \dot{\theta}_2) + 3\omega_0^2(I_1 - I_3)\theta_2 \quad \overset{\mathscr{L}}{\Longrightarrow} \quad \mathbb{T}_{c_2} = -\big[(K_p - C)\Theta_2 + K_d s\Theta_2\big] \tag{19.7}$$

substituting which into Eq. (19.6) and rearranging yields the following input/output relationship mediated by the system's close-loop transfer function:

$$\Theta_2 = \frac{1}{I_2 s^2 + \cancel{C} + (K_p - \cancel{C}) + K_d s}\mathbb{T}_{d_2} = \frac{1/I_2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}\mathbb{T}_d \quad , \quad \zeta \triangleq \frac{K_d}{2}\sqrt{\frac{1}{K_p I_2}} \; , \; \omega_0 \triangleq \sqrt{\frac{K_p}{I_2}} \tag{19.8}$$

where ζ and ω_0 are the damping ratio and undamped natural frequency previously encountered in ACTIVE ATTITUDE CONTROL.

Steady-State Performance

Let there be a constant disturbance torque of magnitude D_2 (step input), $\boldsymbol{\tau}_{d_2} = D_2 H(t)$, with Laplace transform $\mathbb{T}_{d_2} = D_2/s$. The steady-state value of the pitch output subject to the modified PD law in Eq. (19.7) is found by:

$$\theta_{ss} = \lim_{t\to\infty}\theta_2(t) = \lim_{s\to 0} s\theta_2(s) = \lim_{s\to 0}\cancel{s}\frac{1}{I_2 s^2 + K_d s + K_p}\cdot\frac{D_2}{\cancel{s}} = \frac{D_2}{K_p} \tag{19.9}$$

where the final value theorem of Laplace transform, mentioned in ACTIVE ATTITUDE CONTROL, is used. This result implies that the steady-state error in pitch can be reduced by increasing the control gain (which would require more fuel or power), and gain selection can be performed keeping the maximum acceptable θ_{ss} in mind. In order to find the wheel's angular speed required to achieve the desired control torque, $\tau_{c_2}(t) = \dot{h}_s(t) = I_s\dot{\omega}_s$, we have:

$$I_s\big(\omega_s(t) - \omega_{s_0}\big) = \int_0^t \dot{h}_s\, dt \quad \Rightarrow \quad \omega_s = \omega_{s_0} + \frac{\int_0^t \tau_{c_2}\, dt}{I_s} \tag{19.10}$$

Roll/Yaw Control

For simplicity of notation, we replace $\delta\theta_1$ and $\delta\theta_3$ with θ_1 and θ_3, respectively, and considering a stabilization problem in the presence of disturbances, we let $\theta_{1_{ref}} = \theta_{3_{ref}} = 0$, $\theta_1(0) = \dot{\theta}_1(0) = 0$, and $\theta_3(0) = \dot{\theta}_3(0) = 0$. Taking the Laplace transform of the motion equations in Eqs. (19.5a) and (19.5c) results in:

$$I_1 s^2 \Theta_1 + h_s s \Theta_3 + h_s \omega_0 \Theta_1 = \mathcal{T}_{c_1} + \mathcal{T}_{d_1} \tag{19.11a}$$

$$I_3 s^2 \Theta_3 - h_s s \Theta_1 + h_s \omega_0 \Theta_3 = \mathcal{T}_{c_3} + \mathcal{T}_{d_3} \tag{19.11b}$$

Consider the following modified P control laws and their Laplace transforms:

$$\tau_{c_1} = K_p(\cancelto{0}{\theta_{1_{ref}}} - \theta_1) + h_s\omega_0\theta_1 \quad \overset{\mathscr{L}}{\Longrightarrow} \quad \mathcal{T}_{c_1} = -K_p\Theta_1 + h_s\omega_0\Theta_1 \tag{19.12a}$$

$$\tau_{c_3} = -K_r K_p(\cancelto{0}{\theta_{1_{ref}}} - \theta_1) - h_s\dot{\theta}_1 \quad \overset{\mathscr{L}}{\Longrightarrow} \quad \mathcal{T}_{c_3} = K_r K_p\Theta_1 - h_s s\Theta_1 \tag{19.12b}$$

which benefit from using only θ_1 and $\dot{\theta}_1$ measurements of roll angle and rate, hence circumventing the difficulties associated with yaw measurement.

To see one motivation behind such a selection of control laws, consider the closed-loop roll/yaw dynamics, and assume $h_s\dot{\theta}_1 \gg I_3\ddot{\theta}_3$ and $h_s\dot{\theta}_3 \gg I_1\ddot{\theta}_1$ (which are reasonable for a rapidly spinning wheel). Assume, also, no disturbances:

$$h_s\dot{\theta}_3 + \cancel{h_s\omega_0\theta_1} \approx -K_p\theta_1 + \cancel{h_s\omega_0\theta_1} \quad \Rightarrow \quad h_s\dot{\theta}_3 \approx -K_p\frac{h_s\omega_0}{K_r K_p}\theta_3 \quad \Rightarrow \quad \theta_3 \approx \theta_3(0)e^{-\omega_0 t/K_r} \tag{19.13a}$$

$$\cancel{-h_s\dot{\theta}_1} + h_s\omega_0\theta_3 \approx K_r K_p\theta_1 - \cancel{h_s\dot{\theta}_1} \quad \Rightarrow \quad \theta_1 \approx \theta_3(0)\frac{h_s\omega_0}{K_r K_p}e^{-\omega_0 t/K_r} \tag{19.13b}$$

where Eq. (19.13b) is initially substituted into Eq. (19.13a), and the final result of Eq. (19.13a) is substituted back into Eq. (19.13b). Both roll and yaw angles asymptotically reach 0, suggesting *asymptotic stability* in the absence of disturbances.

Returning to the original (disturbed) roll/yaw equations of motion in Eq. (19.11) using the control inputs in Eq. (19.12), and putting the closed-loop system in matrix form yields:

$$\begin{bmatrix} I_1 s^2 + K_p & h_s s \\ -K_r K_p & I_3 s^2 + h_s\omega_0 \end{bmatrix} \begin{bmatrix} \Theta_1 \\ \Theta_3 \end{bmatrix} = \begin{bmatrix} \mathcal{T}_{d_1} \\ \mathcal{T}_{d_3} \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \Theta_1 \\ \Theta_3 \end{bmatrix} = \mathbb{G} \begin{bmatrix} \mathcal{T}_{d_1} \\ \mathcal{T}_{d_3} \end{bmatrix} \tag{19.14}$$

where $\mathbb{G}$ is the system's closed-loop transfer matrix. The characteristic equation is, thus, given by:

$$\det(\mathbb{G}^{-1}) = I_1 I_3 s^4 + (K_p I_3 + h_s\omega_0 I_1)s^2 + K_r K_p h_s s + K_p h_s \omega_0 = 0 \tag{19.15}$$

According to Routh-Hurwitz stability criterion, the system is *not asymptotically stable* owing to the presence of a zero coefficient (of s^3). We can use the following modified PD law instead, with derivative terms added:

$$\tau_{c_1} = K_p(\cancelto{0}{\theta_{1_{ref}}} - \theta_1) + K_d(\cancelto{0}{\dot{\theta}_{1_{ref}}} - \dot{\theta}_1) + h_s\omega_0\theta_1 \quad \overset{\mathscr{L}}{\Longrightarrow} \quad \mathcal{T}_{c_1} = -(K_p + K_d s)\Theta_1 + h_s\omega_0\Theta_1 \tag{19.16a}$$

$$\tau_{c_3} = -K_r K_p(\cancelto{0}{\theta_{1_{ref}}} - \theta_1) - K_r K_d(\cancelto{0}{\dot{\theta}_{1_{ref}}} - \dot{\theta}_1) - h_s\dot{\theta}_1 \quad \overset{\mathscr{L}}{\Longrightarrow} \quad \mathcal{T}_{c_3} = K_r(K_p + K_d s)\Theta_1 - h_s s\Theta_1 \tag{19.16b}$$

which result in the following closed-loop dynamics when substituted into Eq. (19.11):

$$\begin{bmatrix} I_1 s^2 + K_d s + K_p & h_s s \\ -K_r(K_d s + K_p) & I_3 s^2 + h_s \omega_0 \end{bmatrix} \begin{bmatrix} \Theta_1 \\ \Theta_3 \end{bmatrix} = \begin{bmatrix} \Tau_{d_1} \\ \Tau_{d_3} \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \Theta_1 \\ \Theta_3 \end{bmatrix} = \mathbb{H} \begin{bmatrix} \Tau_{d_1} \\ \Tau_{d_3} \end{bmatrix} \tag{19.17}$$

where $\mathbb{H}$ is the system's closed-loop transfer matrix.

Steady-State Performance

Let there be constant disturbance torques of magnitude D_i (step input), $\boldsymbol{\tau}_{d_1} = D_1 H(t)$ and $\boldsymbol{\tau}_{d_3} = D_3 H(t)$, with Laplace transforms $\Tau_{d_1} = D_1/s$ and $\Tau_{d_3} = D_3/s$. The steady-state values of the roll and yaw outputs subject to the modified PD law in Eq. (19.16) are found by:

$$\boldsymbol{\theta}_{ss} = \lim_{t\to\infty} \boldsymbol{\theta}(t) = \lim_{s\to 0} s\Theta(s) = \lim_{s\to 0} \cancel{s}\mathbb{H} \begin{bmatrix} \frac{D_1}{\cancel{s}} \\ \frac{D_3}{\cancel{s}} \end{bmatrix} \tag{19.18}$$

where $\boldsymbol{\theta}(t) \triangleq [\theta_1(t)\ \theta_3(t)]^\intercal$, and $\Theta(s) \triangleq [\Theta_1(s)\ \Theta_3(s)]^\intercal$, and the final value theorem of Laplace transform is used again. We thus have:

$$\boldsymbol{\theta}_{ss} = \lim_{s\to 0} \frac{\begin{bmatrix} I_3 s^2 + h_s \omega_0 & -h_s s \\ K_r(K_d s + K_p) & I_1 s^2 + K_d s + K_p \end{bmatrix}}{(I_1 s^2 + K_d s + K_p)(I_3 s^2 + \omega_0 h_s) + h_s s(K_r K_d s + K_r K_p)} \begin{bmatrix} D_1 \\ D_3 \end{bmatrix} \tag{19.19}$$

which simplifies as follows:

$$\boldsymbol{\theta}_{ss} = \frac{1}{K_p \omega_0 h_s} \begin{bmatrix} \omega_0 h_s & 0 \\ K_r K_p & K_P \end{bmatrix} \begin{bmatrix} D_1 \\ D_3 \end{bmatrix} = \begin{bmatrix} \frac{D_1}{K_p} \\ \frac{K_r D_1}{\omega_0 h_s} + \frac{D_3}{\omega_0 h_s} \end{bmatrix} \tag{19.20}$$

As expected based on our choice of control laws in Eq. (19.16) (using θ_1 and $\dot{\theta}_1$ only), a step disturbance torque about the yaw axis does not affect roll, while that about the roll axis does influence yaw.