Lecture 20

Nonlinear Attitude Control

An introduction to nonlinear attitude control of spacecraft is offered. The state-space representation of the nonlinear system of attitude dynamics is provided, and nonlinear stability analysis tools are employed to assess the suitability of simple control laws involving partial state feedback.

Overview

The linear stability and control tools developed and used thus far are no longer valid if the attitude angles and rates exceed values that are considered to be "small". Initial tumbling of spacecraft upon deployment from their launch vehicle, for example, cannot be expected to be or remain in the linear region on which we have focused to this point. A brief introduction to nonlinear analogues of the stability and control techniques is, therefore, in order.

Recall, from Stability, that a nonlinear system is *not* necessarily representable using a matrix $\boldsymbol{A}$ that linearly maps the state vector to its derivatives. Consider the following representation of a nonlinear system:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}\big(\boldsymbol{x}(t), \boldsymbol{u}(t), t\big) \quad , \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{20.1a}$$

$$\boldsymbol{y}(t) = \boldsymbol{h}\big(\boldsymbol{x}(t), t\big) \tag{20.1b}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^{n\times 1}$ is the state vector, $\boldsymbol{u}(t) \in \mathbb{R}^{m\times 1}$ is the control input vector, and $\boldsymbol{y}(t) \in \mathbb{R}^{l\times 1}$ is the output (measurement) vector. The vector-valued real functions $\boldsymbol{f}$ and $\boldsymbol{h}$ are, in general, nonlinear, and represent the state and measurement models, respectively.

Note: The system in Eq. (20.1) is called "autonomous" if $\boldsymbol{f}$ and $\boldsymbol{h}$ are not explicit functions of time.

Nonlinear Equations of Rotational Motion

Recall, from KINEMATICS, the following relationship between the quaternion representation of attitude and the quaternion rates, as expressed in a body-fixed frame, $\mathscr{F}_B$:

$$\overset{\bullet}{\boldsymbol{\epsilon}}(t) = \frac{1}{2}(\eta\mathbf{1} + \boldsymbol{\epsilon}^{\times})\boldsymbol{\omega} \tag{20.2a}$$

$$\dot{\eta}(t) = -\frac{1}{2}\boldsymbol{\epsilon}^{\mathsf{T}}\boldsymbol{\omega} \tag{20.2b}$$

and recall, from DYNAMICS, Euler's equations of rigid body motion expressed in a body-fixed frame, $\mathscr{F}_B$:

$$\boldsymbol{I}\overset{\bullet}{\boldsymbol{\omega}}(t) + \boldsymbol{\omega}^{\times}(t)\boldsymbol{I}\boldsymbol{\omega}(t) = \boldsymbol{\tau}(t) \quad \Rightarrow \quad \overset{\bullet}{\boldsymbol{\omega}}(t) = \boldsymbol{I}^{-1}\big(-\boldsymbol{\omega}^{\times}(t)\boldsymbol{I}\boldsymbol{\omega}(t) + \boldsymbol{\tau}_c(t) + \boldsymbol{\tau}_d(t)\big) \tag{20.3}$$

where the external torque $\boldsymbol{\tau}$ consists of both control and disturbance torques, $\boldsymbol{\tau}_c$ and $\boldsymbol{\tau}_d$.

The attitude kinematics and dynamics equations in Eqs. (20.2) and (20.3) can be combined to form the following complete set of nonlinear equations of motion:

$$\overset{\bullet}{\boldsymbol{x}}(t) = \boldsymbol{f}\big(\boldsymbol{x}(t), \boldsymbol{u}(t), t\big) \;\; , \;\; \boldsymbol{x} \triangleq \begin{bmatrix} \boldsymbol{\epsilon} \\ \eta \\ \boldsymbol{\omega} \end{bmatrix} \; , \; \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t) \triangleq \begin{bmatrix} \frac{1}{2}(\eta\mathbf{1} + \boldsymbol{\epsilon}^{\times})\boldsymbol{\omega} \\ -\frac{1}{2}\boldsymbol{\epsilon}^{\mathsf{T}}\boldsymbol{\omega} \\ \boldsymbol{I}^{-1}(-\boldsymbol{\omega}^{\times}\boldsymbol{I}\boldsymbol{\omega} + \boldsymbol{\tau}_c + \boldsymbol{\tau}_d) \end{bmatrix} \tag{20.4}$$

Stability Analysis of Rate Feedback

Assume a large initial angular velocity, $\boldsymbol{\omega}_0 \triangleq \boldsymbol{\omega}(0)$ (such as that encountered in initial tumbling), and let the control objective be driving $\boldsymbol{\omega}(t) \to \mathbf{0}$ as $t \to \infty$ using rate feedback only (and disregarding disturbances):

$$\boldsymbol{\tau}_c = -K_d\boldsymbol{\omega}(t) \quad \Rightarrow \quad \overset{\bullet}{\boldsymbol{\omega}} = \boldsymbol{g}(\boldsymbol{\omega}) = \boldsymbol{I}^{-1}(-\boldsymbol{\omega}^{\times}\boldsymbol{I}\boldsymbol{\omega} - K_d\boldsymbol{\omega}) \tag{20.5}$$

where $K_d > 0$, and it is evident that $\boldsymbol{\omega} = \mathbf{0}$ is an equilibrium of this system for it renders $\boldsymbol{g} = \mathbf{0}$, and we need this equilibrium to be *(globally) asymptotically stable*, as defined in STABILITY. Since the control objective, the control input, and the nonlinear function are all in terms of $\boldsymbol{\omega}$ only, there is no need to include the attitude relations presented in Eq. (20.2).

We first show that the spacecraft's angular velocity is bounded by its initial rotational kinetic energy. Let $I_1 \leqslant I_2 \leqslant I_3$ represent the spacecraft's principal moments of inertia. The rotational kinetic energy, T, is bounded as follows:

$$T = \frac{1}{2}\boldsymbol{\omega}^{\times}\boldsymbol{I}\boldsymbol{\omega} \quad \Rightarrow \quad \frac{1}{2}I_1\omega^2(t) \leqslant T(t) \leqslant \frac{1}{2}I_3\omega^2(t) \quad \Rightarrow \quad \omega(t) \leqslant \sqrt{\frac{2T(t)}{I_1}} \leqslant \omega(t)\sqrt{\frac{I_3}{I_1}} \tag{20.6}$$

where $\omega \triangleq |\boldsymbol{\omega}|$. Therefore, since $\omega \leqslant \sqrt{2T/I_1}$, $\boldsymbol{\omega}(t)$ is bounded if $T(t)$ is such.

Note: To see where the first set of inequalities in Eq. (20.6) comes from, assume $\boldsymbol{I}$ is the moment of inertia matrix corresponding to the principal axes frame, for which $T = (I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)/2$. The inequalities follow by evoking the $I_1 \leqslant I_2 \leqslant I_3$ condition.

Consider the rate of change of the rotational kinetic energy:

$$\dot{T} = \boldsymbol{\omega}^{\mathsf{T}} \boldsymbol{I} \mathring{\boldsymbol{\omega}} = \boldsymbol{\omega}^{\mathsf{T}} \cancelto{\mathbf{1}}{\boldsymbol{I}\boldsymbol{I}^{-1}} (-\boldsymbol{\omega}^{\times} \boldsymbol{I} \boldsymbol{\omega} - K_d \boldsymbol{\omega}) = -\cancelto{\mathbf{0}}{\boldsymbol{\omega}^{\mathsf{T}} \boldsymbol{\omega}^{\times}} \boldsymbol{I} \boldsymbol{\omega} - K_d \boldsymbol{\omega}^{\mathsf{T}} \boldsymbol{\omega} \;\;\Rightarrow\;\; \dot{T} = -K_d \omega^2 \leqslant 0 \tag{20.7}$$

where Eq. (20.5) is used to replace $\mathring{\boldsymbol{\omega}}$ and the triple product identity is used for further simplification. Since the energy keeps decreasing as a result of our feedback control, we have:

$$T(t) \leqslant T_0 \;\;\Rightarrow\;\; \omega(t) \leqslant \sqrt{\frac{2T_0}{I_1}} \;\text{ for all } t \geqslant 0 \tag{20.8}$$

where $T_0 \triangleq T(0)$ is the initial energy, and Eq. (20.6) is used. We now establish stability (in the sense of Lyapunov) of the equilibrium $\boldsymbol{\omega} = \mathbf{0}$ using the definition provided in STABILITY: given some $\epsilon > 0$, can we find $\delta > 0$ with $\omega_0 < \delta$ that would guarantee that $\omega(t) < \epsilon$ for all $t \geqslant 0$? To have $\omega(t) < \epsilon$, it would suffice (based on Eq. (20.8)) to have $\sqrt{2T_0/I_1} < \epsilon$, to satisfy which we select $\delta = \epsilon\sqrt{I_1/I_3}$, in order to obtain:

$$\omega_0 < \delta = \epsilon\sqrt{\frac{I_1}{I_3}} \;\;\Rightarrow\;\; \sqrt{\frac{2T_0}{I_1}} \leqslant \omega_0\sqrt{\frac{I_3}{I_1}} < \delta\sqrt{\frac{I_3}{I_1}} = \epsilon \;\;\Rightarrow\;\; \omega(t) \leqslant \sqrt{\frac{2T_0}{I_1}} < \epsilon \tag{20.9}$$

where Eq. (20.6) is used twice. Since such a $\delta > 0$ can be found for *any* $\epsilon > 0$, the equilibrium $\boldsymbol{\omega}_0$ of Eq. (20.5) is *stable*.

To show that $T(t) \to 0$ as $t \to \infty$ (hence $\omega(t) \to 0$ as a consequence, since $\omega \leqslant \sqrt{2T/I_1}$), we assume, by way of contradiction, that $T(t) \geqslant c$ from some $c > 0$. Using Eq. (20.7) and negative of Eq. (20.6) (that is, $-I_3\omega^2/2 \leqslant -T$), we would have:

$$\dot{T} = -K_d\omega^2 \leqslant \frac{-2K_dT}{I_3} \leqslant \frac{-2K_dc}{I_3} \;\;\Rightarrow\;\; T(t) = T_0 + \int_0^t \dot{T}\, d\tau \leqslant T_0 - \frac{2K_dc}{I_3}t \tag{20.10}$$

which would imply that $T(t)$ is at best a linearly decreasing function of time, and could eventually become negative. This would contradict the condition $T(t) > 0$ required by the definition of T in Eq. (20.6) (keeping in mind that $\boldsymbol{I}$ is positive definite), so the assumption that $T(t) \geqslant c$ from some non-zero $c > 0$ cannot hold. Therefore, $T(t) \to 0$ as $t \to \infty$, so does $\boldsymbol{\omega}(t) \to \mathbf{0}$. Since this conclusion can be reached for any arbitrary $\boldsymbol{\omega}_0$, the equilibrium $\boldsymbol{\omega} = \mathbf{0}$ is *globally asymptotically stable*.