Lecture 3

Dynamics

Having studied the geometry of motion, this lesson adds the physical laws of nature, primarily Newton's laws of motion, and attempts to describe rigid body motion in the presence of external forces and torques. The study begins with a point mass, continues on to system of masses, and concludes with continua of point masses that closely resemble a rigid body.

Dynamics of a Point Mass

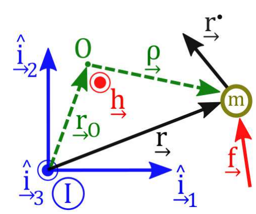

Figure 3.1: Point Mass

Consider a body of mass m, as shown in Figure 3.1, with an infinitesimally small size, relatively speaking, that can be represented by a point mass. Although planets are not particularly small, they could be approximated as point masses in the large scale of the solar system. Consider, also, an inertial frame, $\mathscr{F}_I$.

The relationship between acceleration (2$^{\text{nd}}$ derivative, as seen in $\mathscr{F}_I$, of the position vector of the point mass, $\underrightarrow{r}$) and an external force, $\underrightarrow{f}$, is readily provided by Newton's 2$^{\text{nd}}$ law:

$$\underrightarrow{f} = m\underrightarrow{r}^{\bullet\bullet} \tag{3.1}$$

where $(\cdot)^\bullet$ denotes vector derivative as measured in $\mathscr{F}_I$, and $\underrightarrow{r}$ is the absolute position of the mass, from the origin of $\mathscr{F}_I$, O_I. We also define linear momentum, $\underrightarrow{p}$, and relate it to force as follows:

$$\underrightarrow{p} \triangleq m\underrightarrow{r}^\bullet \quad \Rightarrow \quad \underrightarrow{f} = \underrightarrow{p}^\bullet \tag{3.2}$$

where, once again, $\underrightarrow{r}$ is the position from O_I. Now, consider an arbitrary point, O, which may or may not coincide with O_I. Both force and momentum are independent of O, but angular momentum is not: we define *absolute* angular momentum (or moment of momentum) *about point* O, as follows:

$$\underrightarrow{H}_O \triangleq \underrightarrow{\rho} \times \underrightarrow{p} \quad \Rightarrow \quad \underrightarrow{H}_O = m\underrightarrow{\rho} \times \underrightarrow{r}^\bullet = m\underrightarrow{\rho} \times \left(\underrightarrow{r}_O + \underrightarrow{\rho}\right)^\bullet = m\underrightarrow{\rho} \times \underrightarrow{r}_O^\bullet + m\underrightarrow{\rho} \times \underrightarrow{\rho}^\bullet \tag{3.3}$$

where $\underrightarrow{r}_O$ is the absolute position of O (with respect to O_I), and $\underrightarrow{\rho}$ is the relative position of the point mass with respect to O. Lastly, we define *(relative)* angular momentum *about* O and relate it to absolute angular momentum as follows:

$$\underrightarrow{h}_O \triangleq m\underrightarrow{\rho} \times \underrightarrow{\rho}^\bullet \quad \Rightarrow \quad \underrightarrow{H}_O = m\underrightarrow{\rho} \times \underrightarrow{r}_O^\bullet + \underrightarrow{h}_O \tag{3.4}$$

The distinction between $\underrightarrow{H}_O$ and $\underrightarrow{h}_O$ vanishes if O is inertially fixed (in which case $\underrightarrow{r}_O^{\bullet} = \underrightarrow{0}$), or if $O \equiv O_I$ (in which case $\underrightarrow{\rho} \equiv \underrightarrow{r}$). In this course, we will only deal with one of these special cases, so henceforth, angular momentum as defined in Eq. (3.4) will only be used.

The kinetic energy of a point mass is provided by $T = \frac{1}{2}m\underrightarrow{v} \cdot \underrightarrow{v} = \frac{1}{2}m\underrightarrow{r}^{\bullet} \cdot \underrightarrow{r}^{\bullet}$, and is also a useful quantity when studying motion, such as in a Keplerian orbit with a constant total energy.

We can also express all of these physical quantities in their referential form in $\mathscr{F}_I$:

$$\boldsymbol{f}_I = m\ddot{\boldsymbol{r}}_I \quad , \quad \boldsymbol{p}_I = m\dot{\boldsymbol{r}}_I \quad , \quad \boldsymbol{h}_{O_I} = m\boldsymbol{\rho}_I^{\times}\dot{\boldsymbol{\rho}}_I \quad , \quad T = \frac{1}{2}m\dot{\boldsymbol{r}}_I^{\mathsf{T}}\dot{\boldsymbol{r}}_I$$

Note: If the components of each vector are to be expressed in a non-inertial reference frame, $\mathscr{F}_A$, the referential form of the transport theorem, presented in KINEMATICS and involving the rotation matrix, must be used: for example, for $\mathscr{F}_A$ rotating with an angular velocity of $\underrightarrow{\omega}$ with respect to $\mathscr{F}_I$, we have $\boldsymbol{p}_I = m\dot{\boldsymbol{r}}_I = m\boldsymbol{C}_{IA}\left(\dot{\boldsymbol{r}}_A + \boldsymbol{\omega}_A^{\times}\boldsymbol{r}_A\right)$ for linear momentum.

Dynamics of a System of Point Masses

Consider a collection of N distinct point masses as shown in Figure 3.2, each with mass m_i, and let $\underrightarrow{f}_{ij}$ represent the internal force on m_i exerted by m_j. Also, let $\underrightarrow{f}_i$ denote the external force on m_i. In addition, consider an inertial frame, $\mathscr{F}_I$, and let $\underrightarrow{r}_i$ denote the position vector of each mass with respect to the origin, O_I.

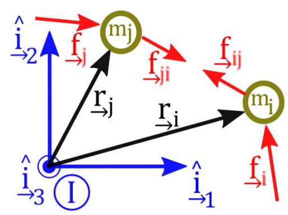

Figure 3.2: System of Point Masses

Applying Newton's 2nd law, first for each mass and then considering all masses, results in:

$$\underrightarrow{f}_i + \sum_{j=1}^{N}\underrightarrow{f}_{ij} = m_i\underrightarrow{r}_i^{\bullet\bullet} \quad \Rightarrow \quad \sum_{i=1}^{N}\underrightarrow{f}_i + \cancelto{\underrightarrow{0}}{\sum_{i=1}^{N}\sum_{j=1}^{N}\underrightarrow{f}_{ij}} = \sum_{i=1}^{N}m_i\underrightarrow{r}_i^{\bullet\bullet} \tag{3.5}$$

where cancellation results from Newton's 3rd law, noting that $\underrightarrow{f}_{ij} = -\underrightarrow{f}_{ji}$.

Definition. The *centre of mass* of a system of particles is located at $\underrightarrow{r}_{\oplus}$, which satisfies the following relationship:

$$\left(\sum_{i=1}^{N}m_i\right)\underrightarrow{r}_{\oplus} = m\underrightarrow{r}_{\oplus} = \sum_{i=1}^{N}m_i\underrightarrow{r}_i \quad \Rightarrow \underrightarrow{r}_{\oplus} \triangleq \frac{\sum_{i=1}^{N}m_i\underrightarrow{r}_i}{m} \tag{3.6}$$

where $m \triangleq \sum_{i=1}^{N}m_i$ is the total mass of the system.

Using Eq. (3.6) and defining $\underrightarrow{f}_{ext} \triangleq \sum_{i=1}^{N}\underrightarrow{f}_i$ as the total external force on the system, Eq. (3.5) can be rewritten as:

$$m\underrightarrow{r}_{\oplus}^{\bullet\bullet} = \underrightarrow{f}_{ext} \tag{3.7}$$

which is analogous to Eq. (3.1) for a single point mass. Defining $\underrightarrow{p} \triangleq \sum_{i=1}^{N}\underrightarrow{p}_i$ as the sum of the point masses' linear momenta, we have:

$$\underrightarrow{p}_i = m_i\underrightarrow{r}_i^{\bullet} \quad \Rightarrow \quad \underrightarrow{p}^{\bullet} = m\underrightarrow{r}_{\oplus}^{\bullet\bullet} = \underrightarrow{f}_{ext} \tag{3.8}$$

which is analogous to Eq. (3.2) for a single point mass. Considering an arbitrary point, O as shown in Figure 3.3, and defining $\underrightarrow{H}_O \triangleq \sum_{i=1}^{N}\underrightarrow{\rho}_i \times \underrightarrow{p}_i$ (where $\underrightarrow{\rho}_i$ is the relative position of the i^{th} mass with respect to O)

as the sum of the point masses' *absolute* angular momenta *about* O, we have:

$$\vec{\boldsymbol{H}}_{O_i} = \vec{\boldsymbol{\rho}}_i \times \vec{\boldsymbol{p}}_i \quad \Rightarrow \quad \vec{\boldsymbol{H}}_O = \sum_{i=1}^{N} m_i \vec{\boldsymbol{\rho}}_i \times \left(\vec{\boldsymbol{r}}_O + \vec{\boldsymbol{\rho}}_i\right)^{\bullet} = \left(\sum_{i=1}^{N} m_i \vec{\boldsymbol{\rho}}_i\right) \times \vec{\boldsymbol{r}}_O^{\bullet} + \sum_{i=1}^{N} m_i \vec{\boldsymbol{\rho}}_i \times \vec{\boldsymbol{\rho}}_i^{\bullet} \tag{3.9}$$

where $\vec{\boldsymbol{r}}_O$ is the absolute position of point O (with respect to O_I).

Definition. The *first moment of mass* of a system of particles *about point* O is defined as:

$$\vec{\boldsymbol{c}}_O \triangleq \sum_{i=1}^{N} m_i \vec{\boldsymbol{\rho}}_i = m \vec{\boldsymbol{\rho}}_{\oplus} \tag{3.10}$$

where $m \triangleq \sum_{i=1}^{N} m_i$ is the total mass of the system, and $\vec{\boldsymbol{\rho}}_{\oplus}$ is the relative position of its centre of mass with respect to O.

Note: The first moment of mass about the centre of mass is zero, because $\vec{\boldsymbol{c}}_{\oplus} = \sum_{i=1}^{N} m_i \left(\vec{\boldsymbol{r}}_i - \vec{\boldsymbol{r}}_{\oplus}\right) = \vec{\boldsymbol{0}}$ based on the definition of centre of mass from Eq. (3.6).

Lastly, we define *(relative)* angular momentum *about* O, and using Eq. (3.10), relate it to absolute angular momentum as follows:

$$\vec{\boldsymbol{h}}_O \triangleq \sum_{i=1}^{N} m_i \vec{\boldsymbol{\rho}}_i \times \vec{\boldsymbol{\rho}}_i^{\bullet} \quad \Rightarrow \quad \vec{\boldsymbol{H}}_O = \vec{\boldsymbol{c}}_O \times \vec{\boldsymbol{r}}_O^{\bullet} + \vec{\boldsymbol{h}}_O \tag{3.11}$$

which is analogous to Eq. (3.4) for a single point mass. Thus, absolute angular momentum consists of contributions from angular momenta of each point mass in the system about O, as well as a contribution from motion of O. The distinction between $\vec{\boldsymbol{H}}_O$ and $\vec{\boldsymbol{h}}_O$ vanishes if O is inertially fixed (in which case $\vec{\boldsymbol{r}}_O^{\bullet} = \vec{\boldsymbol{0}}$), if $O \equiv O_I$ (in which case $\vec{\boldsymbol{\rho}}_i \equiv \vec{\boldsymbol{r}}_i$), or if $O \equiv \oplus$ (in which case $\vec{\boldsymbol{c}}_O = \vec{\boldsymbol{0}}$). Once again, we will only focus on (relative) angular momentum as defined in the left-hand side relationship of Eq. (3.11).

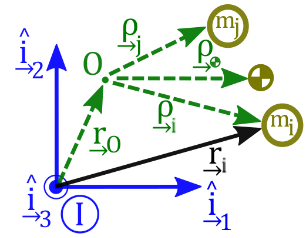

Figure 3.3: Centre of Mass and Arbitrary Point O

We continue our study of system of point masses by differentiating angular momentum with respect to time:

$$\vec{\boldsymbol{h}}_O^{\bullet} = \sum_{i=1}^{N} m_i \left(\cancelto{\vec{\boldsymbol{0}}}{\vec{\boldsymbol{\rho}}_i^{\bullet} \times \vec{\boldsymbol{\rho}}_i^{\bullet}} + \vec{\boldsymbol{\rho}}_i \times \vec{\boldsymbol{\rho}}_i^{\bullet\bullet}\right) = \sum_{i=1}^{N} \vec{\boldsymbol{\rho}}_i \times \left(m_i \vec{\boldsymbol{r}}_i^{\bullet\bullet}\right) - \left(\sum_{i=1}^{N} m_i \vec{\boldsymbol{\rho}}_i\right) \times \vec{\boldsymbol{r}}_O^{\bullet\bullet} \tag{3.12}$$

where $\vec{\boldsymbol{\rho}}_i = \vec{\boldsymbol{r}}_i - \vec{\boldsymbol{r}}_O$ is used. Using Eq. (3.10) and the left-hand side relationship in Eq. (3.5) for $m_i \vec{\boldsymbol{r}}_i^{\bullet\bullet}$, Eq. (3.12) can be rewritten as:

$$\vec{\boldsymbol{h}}_O^{\bullet} = \sum_{i=1}^{N} \vec{\boldsymbol{\rho}}_i \times \vec{\boldsymbol{f}}_i + \sum_{i=1}^{N} \sum_{j=1}^{N} \vec{\boldsymbol{\rho}}_i \times \vec{\boldsymbol{f}}_{ij} - \vec{\boldsymbol{c}}_O \times \vec{\boldsymbol{r}}_O^{\bullet\bullet} \tag{3.13}$$

where the middle term on the right-hand side vanishes if we assume $\vec{\boldsymbol{f}}_{ij}$ acts along $\vec{\boldsymbol{\rho}}_i - \vec{\boldsymbol{\rho}}_j$ (which is the case for gravitational forces, for example) and evoke $\vec{\boldsymbol{f}}_{ji} = -\vec{\boldsymbol{f}}_{ij}$:

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \vec{\boldsymbol{\rho}}_i \times \vec{\boldsymbol{f}}_{ij} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left(\vec{\boldsymbol{\rho}}_i \times \vec{\boldsymbol{f}}_{ij} - \vec{\boldsymbol{\rho}}_j \times \vec{\boldsymbol{f}}_{ij}\right) = \vec{\boldsymbol{0}} \tag{3.14}$$

which is known as the "strong version of Newton's 3rd law". Rearranging Eq. (3.13) upon applying Eq. (3.14) yields the following relationship for external torque *about point* O, $\underset{\to}{\boldsymbol{\tau}}_O$:

$$\underset{\to}{\boldsymbol{\tau}}_O \triangleq \sum_{i=1}^{N} \underset{\to}{\boldsymbol{\rho}}_i \times \underset{\to}{\boldsymbol{f}}_i = \underset{\to}{\boldsymbol{h}}^{\bullet}_O + \underset{\to}{\boldsymbol{c}}_O \times \underset{\to}{\boldsymbol{r}}^{\bullet\bullet}_O \tag{3.15}$$

The second term vanishes for the aforementioned special cases: if O is inertially fixed (in which case $\underset{\to}{\boldsymbol{r}}^{\bullet}_O = \underset{\to}{\boldsymbol{0}}$), or if $O \equiv \oplus$. Therefore, for torque about the centre of mass, we have:

$$\underset{\to}{\boldsymbol{h}}^{\bullet}_{\oplus} = \underset{\to}{\boldsymbol{\tau}}_{\oplus} \tag{3.16}$$

which is the rotational analogue of the force and linear momentum relationship in Eq. (3.8).

System Dynamics in a Rotating Frame

Suppose we have another reference frame, $\mathscr{F}_B$ (such as one attached to a spacecraft's body), rotating with angular velocity $\underset{\to}{\boldsymbol{\omega}}$ with respect to an inertial frame, $\mathscr{F}_I$. In order to express the force/torque relationships studied thus far in terms of time derivatives as measured in $\mathscr{F}_B$, denoted by $(\cdot)^{\circ}$, the transport theorem for first and second derivatives of vectors, presented in KINEMATICS, is required. For example, Eq. (3.8) can be expressed in terms of the dynamics observed in $\mathscr{F}_B$ as follows:

$$\underset{\to}{\boldsymbol{p}}^{\circ} + \underset{\to}{\boldsymbol{\omega}} \times \underset{\to}{\boldsymbol{p}} = m\Big(\underset{\to}{\boldsymbol{r}}^{\circ\circ}_{\oplus} + 2\underset{\to}{\boldsymbol{\omega}} \times \underset{\to}{\boldsymbol{r}}^{\circ}_{\oplus} + \underset{\to}{\boldsymbol{\omega}}^{\circ} \times \underset{\to}{\boldsymbol{r}}_{\oplus} + \underset{\to}{\boldsymbol{\omega}} \times \left(\underset{\to}{\boldsymbol{\omega}} \times \underset{\to}{\boldsymbol{r}}_{\oplus}\right)\Big) = \underset{\to}{\boldsymbol{f}}_{ext} \tag{3.17}$$

In a similar manner, $\underset{\to}{\boldsymbol{\tau}}$ can also be expressed in terms of $\underset{\to}{\boldsymbol{h}}^{\circ}_O$ and $\underset{\to}{\boldsymbol{\omega}}$.

Dynamics of a Rigid Body

We now consider the limiting case of a system of point masses, with $N \to \infty$, where we have a continuum of mass m as opposed to a discrete system. Integrals and differential quantities replace summations and indexed variables. As in Figure 3.4, consider an inertial frame, $\mathscr{F}_I$, and a body-fixed frame, $\mathscr{F}_B$ with arbitrary origin O, and let the position of a differential mass element, dm, with respect to the origin of these frames be denoted by $\underset{\to}{\boldsymbol{r}}$ and $\underset{\to}{\boldsymbol{\rho}}$, respectively.

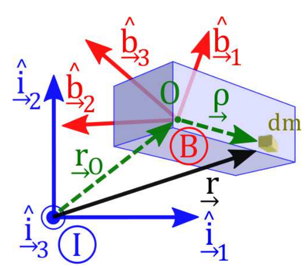

Figure 3.4: Rigid Body with Frame $\mathscr{F}_B$

Definition. A continuum of point masses is a *rigid body* if the distance between any of two points within the body remains fixed. In other words, $\underset{\to}{\boldsymbol{\rho}}^{\circ} = \underset{\to}{\boldsymbol{0}}$, as measured in $\mathscr{F}_B$.

Integrating Eq. (3.1), valid for $dm = \sigma(\underset{\to}{\boldsymbol{\rho}})dV$ (where σ and dV represent density and differential volume, respectively) over the body results in:

$$\underset{\to}{\boldsymbol{p}}^{\bullet} \triangleq \iiint_V \underset{\to}{\boldsymbol{p}}^{\bullet}(\underset{\to}{\boldsymbol{r}})\, dV = \iiint_V \underset{\to}{\boldsymbol{r}}^{\bullet\bullet}\sigma(\underset{\to}{\boldsymbol{\rho}})\, dV \quad\Rightarrow\quad \underset{\to}{\boldsymbol{f}}_{ext} \triangleq \iiint_V \underset{\to}{\boldsymbol{f}}(\underset{\to}{\boldsymbol{r}})\, dV = \int_m \underset{\to}{\boldsymbol{r}}^{\bullet\bullet}\, dm \tag{3.18}$$

Letting $\underset{\to}{\boldsymbol{r}}_{\oplus}$ represent the position of the body's centre of mass, the continuum analogue of Eq. (3.6) is provided by:

$$\left(\int_m dm\right)\underset{\rightarrow}{\boldsymbol{r}}_{\text{◕}} = m\underset{\rightarrow}{\boldsymbol{r}}_{\text{◕}} = \int_m \underset{\rightarrow}{\boldsymbol{r}}\ dm \quad \Rightarrow \quad \underset{\rightarrow}{\boldsymbol{r}}_{\text{◕}} \triangleq \frac{\int_m \underset{\rightarrow}{\boldsymbol{r}}\ dm}{m} \tag{3.19}$$

We also obtain a continuum analogue of the body's first moment of mass *about point* O for later use:

$$\underset{\rightarrow}{\boldsymbol{c}}_O \triangleq \int_m \underset{\rightarrow}{\boldsymbol{\rho}}\ dm = m\underset{\rightarrow}{\boldsymbol{\rho}}_{\text{◕}} \tag{3.20}$$

where $\underset{\rightarrow}{\boldsymbol{\rho}}_{\text{◕}}$ is the relative position of the centre of mass from point O.
Using Eq. (3.19) for centre of mass, Eq. (3.18) can be rewritten as:

$$m\overset{\bullet\bullet}{\underset{\rightarrow}{\boldsymbol{r}}_{\text{◕}}} = \underset{\rightarrow}{\boldsymbol{f}}_{ext} \tag{3.21}$$

This relationship implies that, *in translation*, any continuum of point masses (not just rigid) acts identically to a point mass (located at $\underset{\rightarrow}{\boldsymbol{r}}_{\text{◕}}$) *if* all mass of the body were concentrated at its centre of mass. This will be useful for studying two-body problems involving a spacecraft and a planet, where we will treat the spacecraft as a rigid body with mass m at $\underset{\rightarrow}{\boldsymbol{r}}_{\text{◕}}$.

Assuming $\mathscr{F}_B$ is rotating with respect to $\mathscr{F}_I$ with an angular velocity of $\underset{\rightarrow}{\boldsymbol{\omega}}$, angular momentum in Eq. (3.4) can be integrated over the volume as follows:

$$\underset{\rightarrow}{\boldsymbol{h}}_O \triangleq \iiint_V \underset{\rightarrow}{\boldsymbol{h}}(\underset{\rightarrow}{\boldsymbol{\rho}})\ dV = \iiint_V \underset{\rightarrow}{\boldsymbol{\rho}}\times\overset{\bullet}{\underset{\rightarrow}{\boldsymbol{\rho}}}\sigma(\underset{\rightarrow}{\boldsymbol{\rho}})\ dV = \int_m \underset{\rightarrow}{\boldsymbol{\rho}}\times\big(\cancelto{\underset{\rightarrow}{\mathbf{0}}}{\overset{\circ}{\underset{\rightarrow}{\boldsymbol{\rho}}}} + \underset{\rightarrow}{\boldsymbol{\omega}}\times\underset{\rightarrow}{\boldsymbol{\rho}}\big)\ dm \quad \Rightarrow \underset{\rightarrow}{\boldsymbol{h}}_O = -\int_m \underset{\rightarrow}{\boldsymbol{\rho}}\times\big(\underset{\rightarrow}{\boldsymbol{\rho}}\times\underset{\rightarrow}{\boldsymbol{\omega}}\big)\ dm \triangleq \underset{\rightarrow}{\mathbb{J}}\cdot\underset{\rightarrow}{\boldsymbol{\omega}} \tag{3.22}$$

where $\underset{\rightarrow}{\mathbb{J}}$ is a second order tensor (a "dyadic"), the scalar product of which with a vector is another vector. Expressing the vectors in $\mathscr{F}_B$, namely using $\underset{\rightarrow}{\boldsymbol{h}}_O = \underset{\rightarrow}{\mathscr{F}}_B^{\mathsf{T}}\boldsymbol{h}_O$, $\underset{\rightarrow}{\boldsymbol{\rho}} = \underset{\rightarrow}{\mathscr{F}}_B^{\mathsf{T}}\boldsymbol{\rho}$, and $\underset{\rightarrow}{\boldsymbol{\omega}} = \underset{\rightarrow}{\mathscr{F}}_B^{\mathsf{T}}\boldsymbol{\omega}$ (where we drop the 'B' subscript of the column matrices to avoid clutter), the following referential form of Eq. (3.22) is obtained:

$$\boldsymbol{h}_O = -\int_m \boldsymbol{\rho}^{\times}\boldsymbol{\rho}^{\times}\boldsymbol{\omega}\ dm \triangleq \boldsymbol{J}\boldsymbol{\omega} \tag{3.23}$$

Definition. The *second moment of mass* (or moment of inertia) matrix of a body *about point* O, given some body-fixed frame, $\mathscr{F}_B$, is defined as:

$$\boldsymbol{J} \triangleq -\int_m \boldsymbol{\rho}^{\times}\boldsymbol{\rho}^{\times}\ dm = \iiint_V \left[(\boldsymbol{\rho}^{\mathsf{T}}\boldsymbol{\rho})\mathbf{1} - \boldsymbol{\rho}\boldsymbol{\rho}^{\mathsf{T}}\right]\sigma(\boldsymbol{\rho})\ dV \tag{3.24}$$

where an identity for $\boldsymbol{a}^{\times}\boldsymbol{b}^{\times}$ (for generic $\boldsymbol{a}$ and $\boldsymbol{b}$) from FUNDAMENTALS is used.

Note: If $O \equiv$ ◕, we label $\boldsymbol{J} \equiv \boldsymbol{I}$, since such a moment of inertia matrix provides simplifying consequences.

Moving forward with our study of rigid bodies, based on Eq. (3.15) for a system of point masses and using Eqs. (3.22) and (3.20), we have:

$$\underset{\rightarrow}{\boldsymbol{\tau}}_O \triangleq \iiint_V \underset{\rightarrow}{\boldsymbol{\rho}}\times\underset{\rightarrow}{\boldsymbol{f}}(\underset{\rightarrow}{\boldsymbol{\rho}})\ dV = \overset{\bullet}{\underset{\rightarrow}{\boldsymbol{h}}}_O + \underset{\rightarrow}{\boldsymbol{c}}_O \times \overset{\bullet\bullet}{\underset{\rightarrow}{\boldsymbol{r}}}_O \tag{3.25}$$

Focusing on the case with $O \equiv$ ◕ (for which $\underset{\rightarrow}{\boldsymbol{c}}_{\text{◕}} = \underset{\rightarrow}{\mathbf{0}}$), Eq. (3.25) can also be expressed in terms of the motion observed in $\mathscr{F}_B$, rotating at $\underrightarrow{\boldsymbol{\omega}}$ with respect to $\mathscr{F}_I$:

$$\underrightarrow{\boldsymbol{h}}^{\bullet}_{\odot} = \underrightarrow{\boldsymbol{h}}^{\circ}_{\odot} + \underrightarrow{\boldsymbol{\omega}} \times \underrightarrow{\boldsymbol{h}}_{\odot} = \underrightarrow{\boldsymbol{\tau}}_{\odot} \tag{3.26}$$

where the transport theorem from FUNDAMENTALS is used. Lastly, resolving all vectors in $\mathscr{F}_B$ as before, and noting that $\boldsymbol{h}_{\odot} = \boldsymbol{I}\boldsymbol{\omega}$ from Eq. (3.23), "Euler's rigid body equations" are obtained from the referential form of Eq. (3.26):

$$\boldsymbol{I}\overset{\bullet}{\boldsymbol{\omega}} + \boldsymbol{\omega}^{\times}\boldsymbol{I}\boldsymbol{\omega} = \boldsymbol{\tau}_{\odot} \tag{3.27}$$

Together with Poisson's kinematical equation from KINEMATICS, $\overset{\bullet}{\boldsymbol{C}}_{BI} = -\boldsymbol{\omega}^{\times}\boldsymbol{C}_{BI}$, Eq. (3.27) fully describes the *rotational motion* of a rigid body, and the two sets of equations can be integrated as a coupled system in order to determine the attitude of the body over time.

Properties of Moment of Inertia Matrix

Second moment of mass (or moment of inertia) matrix of a body *about point* O, $\boldsymbol{J}$ as defined in Eq. (3.24), depends on the reference frame, $\mathscr{F}_B$, but it has rotational and translational transformation properties that permit one to obtain the new $\boldsymbol{J}$ resulting from rotating or translating the reference frame. Regardless of the reference frame choice, $\boldsymbol{J}$ is always symmetric ($\boldsymbol{J} = \boldsymbol{J}^{\mathsf{T}}$) and positive-definite ($\boldsymbol{x}^{\mathsf{T}}\boldsymbol{J}\boldsymbol{x} > 0$ for all $\boldsymbol{x} \neq \boldsymbol{0}$).

Rotational Transformation

Consider two body-fixed frames, $\mathscr{F}_A$ and $\mathscr{F}_B$, that share the same origin, O, but are rotated relative to each other, as shown in Figure 3.5. Let $\boldsymbol{C}_{BA}$ represent the rotation matrix from the former to the latter, and let $\boldsymbol{J}_A$ and $\boldsymbol{J}_B$ denote the moment of inertia matrices corresponding to each frame.

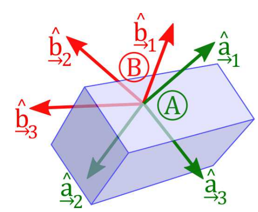

Figure 3.5: Rotation

The relative position of each point within the body with respect to O can be expressed in both frames as:

$$\underrightarrow{\boldsymbol{\rho}} = \underrightarrow{\mathscr{F}}_A^{\mathsf{T}}\boldsymbol{\rho}_A = \underrightarrow{\mathscr{F}}_B^{\mathsf{T}}\boldsymbol{\rho}_B \tag{3.28}$$

using which together with Eq. (3.24) for moment of inertia gives:

$$\boldsymbol{J}_B = -\int_m \boldsymbol{\rho}_B^{\times}\boldsymbol{\rho}_B^{\times}\, dm = -\int_m \left(\boldsymbol{C}_{BA}\boldsymbol{\rho}_A\right)^{\times}\left(\boldsymbol{C}_{BA}\boldsymbol{\rho}_A\right)^{\times}\, dm \tag{3.29}$$

but $\left(\boldsymbol{C}_{BA}\boldsymbol{\rho}_A\right)^{\times} = \boldsymbol{C}_{BA}\boldsymbol{\rho}_A^{\times}\boldsymbol{C}_{AB}$ from KINEMATICS, substituting which into Eq. (3.29) yields:

$$\boldsymbol{J}_B = -\int_m \boldsymbol{C}_{BA}\boldsymbol{\rho}_A^{\times}\overbrace{\cancel{\boldsymbol{C}_{AB}\boldsymbol{C}_{BA}}}^{\boldsymbol{1}}\boldsymbol{\rho}_A^{\times}\boldsymbol{C}_{AB}\, dm = \boldsymbol{C}_{BA}\left(-\int_m \boldsymbol{\rho}_A^{\times}\boldsymbol{\rho}_A^{\times}\, dm\right)\boldsymbol{C}_{AB} \tag{3.30}$$

where orthonormality of a rotation matrix is used. We thus have the following rotational transformation for moment of inertia:

$$\boldsymbol{J}_B = \boldsymbol{C}_{BA}\boldsymbol{J}_A\boldsymbol{C}_{AB} \tag{3.31}$$

Note: In fact, this relationship holds for any dyadic, and is a generalization of $\boldsymbol{u}_B = \boldsymbol{C}_{BA}\boldsymbol{u}_A$ for column matrices.

Translational Transformation

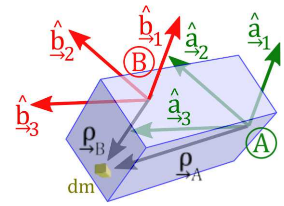

Figure 3.6: Translation

Consider two body-fixed frames, $\mathscr{F}_A$ and $\mathscr{F}_B$, that have their bases vectors oriented parallel in pairs, but with O_B translated to a relative position of $\underset{\rightarrow}{\boldsymbol{r}}_{BA}$ from O_A, as in Fig. 3.6. Let $\underset{\rightarrow}{\boldsymbol{\rho}}_A$ and $\underset{\rightarrow}{\boldsymbol{\rho}}_B$ represent the relative position of each point in the body with respect to O_A and O_B, respectively, such that $\underset{\rightarrow}{\boldsymbol{\rho}}_B = \underset{\rightarrow}{\boldsymbol{\rho}}_A - \underset{\rightarrow}{\boldsymbol{r}}_{BA}$.

Using Eq. (3.24), we have:

$$\boldsymbol{J}_B = -\int\limits_m \boldsymbol{\rho}_B^\times \boldsymbol{\rho}_B^\times \, dm = -\int\limits_m \boldsymbol{\rho}_A^\times \boldsymbol{\rho}_A^\times \, dm + \int\limits_m \boldsymbol{r}_{BA}^\times \boldsymbol{\rho}_A^\times \, dm + \int\limits_m \boldsymbol{\rho}_A^\times \boldsymbol{r}_{BA}^\times \, dm - \int\limits_m \boldsymbol{r}_{BA}^\times \boldsymbol{r}_{BA}^\times \, dm \tag{3.32}$$

which, using Eq. (3.20) for first moment of mass *about point* O_A, can be rewritten as:

$$\boldsymbol{J}_B = \boldsymbol{J}_A + \boldsymbol{r}_{BA}^\times \boldsymbol{c}_{O_A}^\times + \boldsymbol{c}_{O_A}^\times \boldsymbol{r}_{BA}^\times - m \boldsymbol{r}_{BA}^\times \boldsymbol{r}_{BA}^\times \tag{3.33}$$

which provides the translational transformation for moment of inertia, namely the “parallel axis theorem”.

Note: If $O_A \equiv$ ◕ ($\mathscr{F}_A$ placed at the centre of mass), then $\boldsymbol{c}_{O_A} = \boldsymbol{0}$, $\boldsymbol{J}_A \equiv \boldsymbol{I}_A$, and Eq. (3.33) reduces to:

$$\boldsymbol{J}_B = \boldsymbol{I}_A - m \boldsymbol{r}_{BA}^\times \boldsymbol{r}_{BA}^\times \tag{3.34}$$

which describes the changes in moment of inertia when the reference point is off-set from the centre of mass.

Diagonalization

Since $\boldsymbol{J}$ is a symmetric matrix, the moment of inertia matrix corresponding to any body-fixed frame, $\mathscr{F}_B$, can be diagonalized as $\boldsymbol{J}_B = \boldsymbol{E}\boldsymbol{\Lambda}\boldsymbol{E}^{-1}$, where $\boldsymbol{\Lambda}$ is a diagonal matrix of eigenvalues, each of which is called a “principal moment of inertia”; and $\boldsymbol{E}$ is a matrix with eigenvectors as its columns, each of which describes the coordinates (in $\mathscr{F}_B$) of the basis vectors that define the so-called “principal axes frame”. We call $\boldsymbol{\Lambda} \equiv \boldsymbol{J}_P$ the diagonal moment of inertia matrix corresponding to the principal axes frame, and using Eq. (3.31) to move from $\mathscr{F}_B$ to $\mathscr{F}_P$, we have:

$$\boldsymbol{E}^{-1}\boldsymbol{J}_B\boldsymbol{E} = \boldsymbol{\Lambda} \equiv \boldsymbol{J}_P = \boldsymbol{C}_{PB}\boldsymbol{J}_B\boldsymbol{C}_{BP} \tag{3.35}$$

which shows that $\boldsymbol{E}$ is, in fact, a rotation matrix. When studying rigid body motion, choosing $\mathscr{F}_P$ with its origin at the centre of mass results in some simplifications: for example, owing to the diagonal nature of $\boldsymbol{I}$, Euler's equation in Eq. (3.27) can be written as three simple scalar equations, each corresponding to one component of torque:

$$I_1\dot{\omega}_1 + \big(I_3 - I_2\big)\omega_2\omega_3 = \tau_1 \tag{3.36a}$$

$$I_2\dot{\omega}_2 + \big(I_1 - I_3\big)\omega_1\omega_3 = \tau_2 \tag{3.36b}$$

$$I_3\dot{\omega}_3 + \big(I_2 - I_1\big)\omega_1\omega_2 = \tau_3 \tag{3.36c}$$

where I_1, I_2, and I_3 are the principal moments of inertia.