# Lecture 6

# Orbital Perturbations

Thus far, we have only looked at the two-body motion resulting from the gravitational forces between two bodies, for which there exists an analytic solution that describes an ideal Keplerian orbit. We will now consider a more realistic model that accounts for perturbations resulting from external sources, and attempt to provide a more accurate description of the resulting motion.

## Overview

From Orbital Mechanics, we know that an ideal Keplerian orbit is described by:

$$\underset{\rightarrow}{\boldsymbol{r}}^{\bullet\bullet} = \frac{-\mu}{r^3}\underset{\rightarrow}{\boldsymbol{r}} \tag{6.1}$$

and can be uniquely specified using a set of constant orbital parameters; however, external disturbance sources cause deviations from the ideal orbit that result in time-varying orbital parameters. A real perturbed orbit can be, in general, represented by:

$$\underset{\rightarrow}{\boldsymbol{r}}^{\bullet\bullet} = \frac{-\mu}{r^3}\underset{\rightarrow}{\boldsymbol{r}} + \underset{\rightarrow}{\boldsymbol{a}}_p \tag{6.2}$$

where $\underset{\rightarrow}{\boldsymbol{a}}_p$ is the "perturbation acceleration" (perturbation force per unit mass of the body). Examples of perturbation sources include: non-spherical shape of Earth, invalidating the point mass approximation; gravitational effects of other bodies, especially Moon and Sun (affecting the Earth-orbiting spacecraft); solar radiation pressure, resulting from photons' transfer of momentum; and atmospheric drag, depending on the shape and aerodynamic characteristics of the orbiting body.

The approaches used for studying perturbations can be broadly categorized as follows:

- special perturbations: numerical approaches that are dependent on the initial conditions (ICs)

- general perturbations: analytic approaches with closed-form solutions

## Cowell's Method

This is a special perturbations approach that directly integrates the equations of motion in Eq. (6.2). To this end, $\underset{\rightarrow}{r}$ and $\underset{\rightarrow}{v}$ are expressed in $\mathscr{F}_G$, and the 2nd order ODE of Eq. (6.2) is written as a set of two 1st order ODEs:

$$\begin{bmatrix} \overset{\bullet}{\boldsymbol{r}}_G \\ \overset{\bullet}{\boldsymbol{v}}_G \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_G \\ \dfrac{-\mu}{\left(\boldsymbol{r}_G^{\intercal}\boldsymbol{r}_G\right)^{3/2}}\boldsymbol{r}_G + \boldsymbol{a}_{p_G} \end{bmatrix} \tag{6.3}$$

which can be integrated numerically using various techniques, such as a 4th order Runge-Kutta (RK4) integration scheme. This method is simple and straightforward, and works for any type of $\underset{\rightarrow}{a}_p$; however, it is also computationally expensive, since it requires a small step size, and becomes inaccurate for long-term numerical studies.

## Encke's Method

This method, another special perturbations approach, integrates the ODEs describing the orbit's "deviations" over time. Considering an osculating (ideal) Keplerian orbit, $\underset{\rightarrow}{\boldsymbol{\rho}}(t)$, we have:

$$\underset{\rightarrow}{\boldsymbol{\rho}}^{\bullet\bullet} = \frac{-\mu}{\rho^3}\underset{\rightarrow}{\boldsymbol{\rho}} \;\;,\;\; \underset{\rightarrow}{\boldsymbol{\rho}}(0) = \underset{\rightarrow}{r}_0 \;\;,\;\; \underset{\rightarrow}{\boldsymbol{\rho}}^{\bullet}(0) = \underset{\rightarrow}{v}_0 \tag{6.4}$$

where the same ICs as those of the true (perturbed) orbit are used, since the ideal orbit is considered to be tangent to the true on at time 0. Now, defining $\delta\underset{\rightarrow}{r} \triangleq \underset{\rightarrow}{r} - \underset{\rightarrow}{\boldsymbol{\rho}}$ as the true orbit's deviation from the osculating orbit, we have:

$$\delta\underset{\rightarrow}{r}^{\bullet\bullet} = \underset{\rightarrow}{r}^{\bullet\bullet} - \underset{\rightarrow}{\boldsymbol{\rho}}^{\bullet\bullet} = \frac{-\mu}{r^3}\underset{\rightarrow}{r} + \underset{\rightarrow}{a}_p + \frac{\mu}{\rho^3}\underset{\rightarrow}{\boldsymbol{\rho}} = \frac{-\mu}{\rho^3}\left[\delta\underset{\rightarrow}{r} - \left(1 - \frac{\rho^3}{r^3}\right)\underset{\rightarrow}{r}\right] + \underset{\rightarrow}{a}_p \;\;,\;\; \delta\underset{\rightarrow}{r}(0) = \underset{\rightarrow}{0} \;\;,\;\; \delta\underset{\rightarrow}{r}^{\bullet}(0) = \underset{\rightarrow}{0} \tag{6.5}$$

where Eq. (6.4) is used. Since $\delta\underset{\rightarrow}{r}$ changes much more slowly than $\underset{\rightarrow}{r}$, this method can use a larger step size than Cowell's method, and consequently can be less computationally demanding.

*Note*: The term $1 - \rho^3/r^3 \to 0$ for small $\delta r$, which may lead to loss of precision resulting from direct integration of Eq. (6.5). To circumvent this issue, we can define $2q \triangleq 1 - r^2/\rho^2$, so that $\rho^3/r^3 = (1-2q)^{-3/2}$. Then, $1 - \rho^3/r^3$ could be expanded using Taylor series as:

$$1 - \frac{\rho^3}{r^3} = 1 - (1-2q)^{-3/2} \approx 1 - \left(1 - 3q + \frac{3\times 5}{2!}q^2\right) = 3q - \frac{15}{2}q^2 \tag{6.6}$$

which converges rapidly for small $q$, making it suitable for replacement in Eq. (6.5).

While using this method, when $\delta\underset{\rightarrow}{r}$ grows beyond a user-specified tolerance, the osculating orbit is typically updated by setting the ICs of Eq. (6.5) using the true orbit's ICs at the time. This is called "rectification".

## Gauss' Variational Equations

These equations provide analytic expressions for how orbital parameters vary over time, so this is a general perturbations method. We will only look at $\dot{a} = da/dt$ as an example, but similar approaches could be used for the other parameters as well.

Recall, from ORBITAL MECHANICS, the following definition and relationship for orbital specific energy, which can be differentiated with respect to time as shown:

$$\epsilon \triangleq \frac{\underset{\to}{\boldsymbol{v}} \cdot \underset{\to}{\boldsymbol{v}}}{2} - \frac{\mu}{r} = \frac{-\mu}{2a} \quad \Rightarrow \quad \frac{d\epsilon}{dt} = \frac{\mu}{2a^2}\frac{da}{dt} = \cancelto{\underset{\to}{\boldsymbol{r}}^{\bullet}}{\underset{\to}{\boldsymbol{v}}} \cdot \cancelto{\underset{\to}{\boldsymbol{r}}^{\bullet\bullet}}{\underset{\to}{\boldsymbol{v}}^{\bullet}} + \frac{\mu}{r^2}\cancelto{(\underset{\to}{\boldsymbol{r}} \cdot \underset{\to}{\boldsymbol{r}}^{\bullet})/r}{\dot{r}} \tag{6.7}$$

rearranging which for $\dot{a}$ yields, upon substituting the perturbed acceleration from Eq. (6.2):

$$\frac{da}{dt} = \frac{2a^2}{\mu}\frac{d\epsilon}{dt} = \frac{2a^2}{\mu}\left[\cancel{\underset{\to}{\boldsymbol{r}}^{\bullet} \cdot \frac{-\mu}{r^3}\underset{\to}{\boldsymbol{r}}} + \underset{\to}{\boldsymbol{r}}^{\bullet} \cdot \underset{\to}{\boldsymbol{a}}_p + \cancel{\underset{\to}{\boldsymbol{r}}^{\bullet} \cdot \frac{\mu}{r^3}\underset{\to}{\boldsymbol{r}}}\right] \quad \Rightarrow \quad \dot{a} = \frac{2a^2}{\mu}\underset{\to}{\boldsymbol{r}}^{\bullet} \cdot \underset{\to}{\boldsymbol{a}}_p \tag{6.8}$$

Consider the orbiting frame, $\mathscr{F}_O$, with its origin on the primary and its 1-axis pointing to the orbiting body at all times, as shown in Figure 6.1. We have, from ORBITAL MECHANICS and as shown in Figure 6.1 :

$$\underset{\to}{\boldsymbol{r}} = \underset{\to}{\mathscr{F}}_O^{\mathsf{T}} \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} \quad , \quad \underset{\to}{\boldsymbol{r}}^{\bullet} = \underset{\to}{\mathscr{F}}_O^{\mathsf{T}} \begin{bmatrix} \dot{r} \\ r\dot{\theta} \\ 0 \end{bmatrix} \quad , \quad \text{let} \;\; \underset{\sim}{\boldsymbol{a}}_{\mathrm{p}} = \underset{\sim}{\mathscr{F}}_{\mathrm{O}}^{\mathsf{T}} \begin{bmatrix} a_r \\ a_\theta \\ a_z \end{bmatrix} \tag{6.9}$$

using which Eq. (6.8) can be rewritten as:

$$\dot{a} = \frac{2a^2}{\mu}\left(\dot{r}a_r + r\dot{\theta}a_\theta\right) \tag{6.10}$$

We would also like to have a relationship for $\dot{a}$ in terms of orbital elements, as they are more common and convenient to use. To this end, the $\dot{r}$ and $\dot{\theta}$ relationships from ORBITAL MECHANICS (which were, in turn, based on $h = r^2\dot{\theta}$ and the time-derivative of the polar equation of an orbit, respectively) are used to rewrite Eq. (6.10) as:

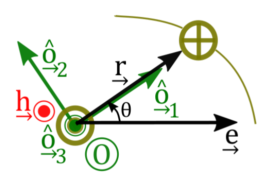


Figure 6.1: Orbiting

$$\dot{a} = \frac{da}{dt} = \frac{2a^2}{\sqrt{\mu a(1-e^2)}}\Big[e\sin(\theta)a_r + \big(1 + e\cos(\theta)\big)a_\theta\Big] \tag{6.11}$$

*Note*: Refer to Section 7.7 of *Spacecraft Dynamics and Control: an Introduction* for the rest of Gauss' variational equations, describing the rate of change of the other orbital parameters.

In addition to perturbation studies in the subsequent sections, another practical example of where Gauss' variational equations could be used is in low-thrust trajectory analysis that involves small continuous thrusting forces as opposed to large impulsive ones. In this case, the force per unit mass terms ($a_x$, $a_y$ and $a_z$) are replaced by the components of the applied thrust vector.

## Perturbations due to a Non-Spherical Primary

Earth is not a perfect sphere, and as a result, perturbations result in geocentric spacecraft's orbits from the differences between the gravitational forces applied by the various parts of Earth's asymmetric body. This perturbation source is also known as $J_2$ *perturbation*.

From Newton's law of gravitation for an orbiting body of mass $m$ and a primary of mass $M$, we have:

$$\underset{\to}{\boldsymbol{f}} = \frac{-GMm}{r^3}\underset{\to}{\boldsymbol{r}} \;\;\Rightarrow\;\; \underset{\to}{\boldsymbol{a}} = \frac{-GM}{r^3}\underset{\to}{\boldsymbol{r}} = \frac{-\mu}{r^3}\underset{\to}{\boldsymbol{r}} = \underset{\to}{\boldsymbol{\nabla}}\phi(\underset{\to}{\boldsymbol{r}}) \tag{6.12}$$

where $\phi(\underset{\to}{\boldsymbol{r}}) \triangleq \mu/r$ is a scalar potential function, the gradient of which is force per unit mass. We know such a potential function exists because the gravitational force is a conservative one.

Instead of treating the primary body (Earth, for example) as a point mass, which we can no longer do because of its asymmetry, we consider it as a collection of differential mass elements, as depicted in Figure 6.2. Adopting a body-fixed frame, $\mathscr{F}_B$, with its origin at the centre of mass, we obtain the body's total potential function, at a spatial position $\underset{\to}{\boldsymbol{r}}$, by integrating over the differential potential of each mass element, located at $\underset{\to}{\boldsymbol{\rho}}$ relative to the centre of mass:

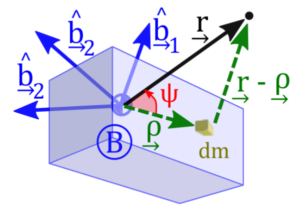


Figure 6.2: Potential Field of a Continuum

$$d\phi(\underset{\to}{\boldsymbol{r}}) = \frac{G\sigma(\underset{\to}{\boldsymbol{\rho}})\,dV(\underset{\to}{\boldsymbol{\rho}})}{|\underset{\to}{\boldsymbol{r}} - \underset{\to}{\boldsymbol{\rho}}|} \;\;\Rightarrow\;\; \phi(\underset{\to}{\boldsymbol{r}}) = \iiint_V \frac{G\sigma(\underset{\to}{\boldsymbol{\rho}})\,dV(\underset{\to}{\boldsymbol{\rho}})}{|\underset{\to}{\boldsymbol{r}} - \underset{\to}{\boldsymbol{\rho}}|} = \frac{G}{r}\iiint_V \frac{\sigma(\underset{\to}{\boldsymbol{\rho}})\,dV(\underset{\to}{\boldsymbol{\rho}})}{\sqrt{1 - 2\frac{\rho}{r}\cos(\psi) + \left(\frac{\rho}{r}\right)^2}} \tag{6.13}$$

where the cosine law is used, $\psi$ is the angle between $\underset{\to}{\boldsymbol{r}}$ and $\underset{\to}{\boldsymbol{\rho}}$, and $\sigma(\underset{\to}{\boldsymbol{\rho}})$ is the density that could vary over the body. We now use power series to expand the reciprocal of the square root term, and obtain:

$$\phi(\underset{\to}{\boldsymbol{r}}) = \frac{G}{r}\sum_{n=0}^{\infty}\iiint_V \sigma(\underset{\to}{\boldsymbol{\rho}})\left(\frac{\rho}{r}\right)^n P_n\big(\cos(\psi)\big)\,dV \tag{6.14}$$

where $P_n$ are Legendre polynomials, with $P_0\big(\cos(\psi)\big) = 1$ , $P_1\big(\cos(\psi)\big) = \cos(\psi)$, etc. Pulling out the first two terms of the summation and substituting these Legendre coefficients yields, upon replacing $\cos(\psi) = (\underset{\to}{\boldsymbol{\rho}} \cdot \underset{\to}{\boldsymbol{r}})/(\rho r)$, the following:

$$\phi(\underset{\to}{\boldsymbol{r}}) = \cancelto{\frac{GM}{r}}{\frac{G}{r}\iiint_V \sigma(\underset{\to}{\boldsymbol{\rho}})\,dV} + \frac{G}{r^3}\underset{\to}{\boldsymbol{r}} \cdot \cancelto{\underset{\to}{\boldsymbol{0}}}{\iiint_V \sigma(\underset{\to}{\boldsymbol{\rho}})\underset{\to}{\boldsymbol{\rho}}\,dV} + \frac{G}{r}\sum_{n=2}^{\infty}\iiint_V \sigma(\underset{\to}{\boldsymbol{\rho}})\left(\frac{\rho}{r}\right)^n P_n\big(\cos(\psi)\big)\,dV \tag{6.15}$$

where the first term is the potential associated with a point mass, $M$, and the volume integral of the second term vanishes by definition of centre of mass, since $O_B \equiv$ ◕. From Eq. (6.15), we observe that the gravitational perturbations result from the third term, which we call the perturbing potential, $\phi_p(\underset{\to}{\boldsymbol{r}})$. It can be shown, using spherical coordinates, that for a body that is axisymmetric about its 3-axis (for which the azimuth angle no longer matters because of symmetry), the last term of Eq. (6.15) reduces to:

$$\phi_p = \frac{-GM}{r}\sum_{n=2}^{\infty} J_n\left(\frac{R}{r}\right)^n P_n\big(\sin(\delta)\big) \tag{6.16}$$

where $R$ is the body's radius (taken to be the equatorial radius for Earth), and $\delta$ is the elevation angle from the 1,2-plane (the equatorial plane for Earth). The variables $J_n$ are known as *zonal harmonic coefficients*, and are functions of the Legendre polynomials, mass density, and the spherical coordinates used. In practice, they are determined experimentally.

*Note*: Refer to Section 7.3 of *Spacecraft Dynamics and Control: an Introduction* for details on how to arrive at Eq. (6.16) using spherical harmonics.

For Earth, $J_2 \approx 1.083 \times 10^{-3}$ and $J_3 \approx -2.53 \times 10^{-6}$. The former results from Earth's oblate spheroid shape, while the latter accounts for its pear-like characteristics. Using only $J_2$ as the dominant term and noting that $P_2\big(\sin(\delta)\big) = (3/2)\sin^2(\delta) - 1/2$, Earth's total gravitational potential is given by:

$$\phi \approx \frac{\mu_\oplus}{r} - \frac{\mu_\oplus}{r^3} J_2 R_\oplus^2 \left(\frac{3}{2}\sin^2(\delta) - \frac{1}{2}\right) \tag{6.17}$$

where the second term is responsible for the so-called $J_2$ perturbation.

Lastly, the perturbation acceleration (force per unit mass) can be obtained via $\underset{\rightarrow}{\boldsymbol{a}}_p = \underset{\rightarrow}{\boldsymbol{\nabla}} \phi_p(\underset{\rightarrow}{\boldsymbol{r}})$. Upon taking the gradient in cylindrical coordinates and expressing each basis vector using orbital elements, Earth's perturbation force components per mass (computed at a distance of $r$ from its centre of mass) are given by:

$$a_r = \frac{\partial \phi}{\partial r} = C\Big(3\sin^2(i)\sin^2(\omega+\theta) - 1\Big) \tag{6.18a}$$

$$a_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -C\sin^2(i)\sin\Big(2(\omega+\theta)\Big) \tag{6.18b}$$

$$a_z = \frac{\partial \phi}{\partial z} = -C\sin(2i)\sin(\omega+\theta) \tag{6.18c}$$

where $C \triangleq \dfrac{3\mu_\oplus J_2 R_\oplus^2}{2r^4}$, with $J_2 \approx 1.083 \times 10^{-3}$ for Earth.

## Effects of $J_2$ Perturbation on Orbital Parameters

In general, the effects of perturbations on orbital parameters can be studied using Gauss' variational equations, in which the perturbation forces per unit mass corresponding to the pertinent source are substituted. The resulting variations can be categorized as follows:

- secular variations: long-term changes that result in an increasing off-set of the parameters
- periodic variations: short-term oscillations that result in no net change over many orbits

Let us focus on $J_2$ perturbations, for which the corresponding force per unit mass components are given by Eq. (6.18). Denoting secular variations with $\langle\dot{\cdot}\rangle \triangleq \Delta(\cdot)/T$ to imply an average of the variations over an orbital period, we have the following secular variations resulting from Earth's $J_2$ effects:

$$\langle\dot{\Omega}\rangle = \frac{-3J_2R_\oplus^2}{2(1-e^2)^2}\sqrt{\frac{\mu_\oplus}{a^7}}\cos(i) \tag{6.19a}$$

$$\langle\dot{\omega}\rangle = \frac{3J_2R_\oplus^2}{4(1-e^2)^2}\sqrt{\frac{\mu_\oplus}{a^7}}\left(5\cos^2(i) - 1\right) \tag{6.19b}$$

$$\langle\dot{a}\rangle = \langle\dot{e}\rangle = \langle\dot{i}\rangle = 0 \tag{6.19c}$$

To see where these relationships come from, let us derive Eq. (6.19a). The respective Gauss' variational equation, derived in pages 174-176 of *Spacecraft Dynamics and Control: an Introduction*, is given by:

$$\frac{d\Omega}{dt} = \sqrt{\frac{a\left(1-e^2\right)}{\mu}}\frac{\sin(\omega+\theta)}{\sin(i)\left(1+e\cos(\theta)\right)}a_z \tag{6.20}$$

Also, using chain rule and $\dot{\theta}$ from ORBIT DESCRIPTION AND DETERMINATION, we can obtain the derivative with respect to the true anomaly, which facilitates integration over an orbit:

$$\frac{d\Omega}{d\theta} = \frac{1}{\dot{\theta}}\frac{d\Omega}{dt} = \frac{r^2}{\sqrt{\mu a\left(1-e^2\right)}}\frac{d\Omega}{dt} \tag{6.21}$$

Substituting the relevant component of the $J_2$ force given by Eq. (6.18c) into Eq. (6.20), substituting the result into Eq. (6.21), and using trigonometric integrals to simplify the expression eventually yields:

$$\frac{d\Omega}{d\theta} = \frac{-3J_2R_\oplus^2}{a^2\left(1-e^2\right)^2}\cos(i)\sin^2(\omega+\theta)\left[1+e\cos(\theta)\right] \tag{6.22}$$

integrating which from $\theta = 0$ to $\theta = 2\pi$ provides an expression for the total variation in $\Omega$ over an orbit:

$$\Delta\Omega = \int_0^{2\pi}\frac{d\Omega}{d\theta}\,d\theta = \frac{-3\pi J_2R_\oplus^2}{a^2\left(1-e^2\right)^2}\cos(i) \tag{6.23}$$

which is, lastly, divided by $T = 2\pi\sqrt{a^3/\mu}$ to yield $\langle\dot{\Omega}\rangle = \Delta\Omega/T$ as given by Eq. (6.19a).

## Design based on $J_2$ Perturbation

Although disturbances such as $J_2$ effects are typically undesirable, they can sometimes be exploited for design purposes. Two types of orbits that are designed with $J_2$ perturbations in mind are now considered.

#### Sun-synchronous Orbits

Given some $a$ and $e$, one can select $i$ such that $\langle\dot{\Omega}\rangle = 2\pi\ \mathrm{rad/yr}$, based on Eq. (6.19a). Using this approach, the resulting orbital plane rotates on the reference frame (the ascending node moves) at the same rate as Earth rotates around Sun. As a result, the Sun-Earth-ascending-node angle remains the same at all times, as illustrated in Figure 6.3. Applications include placing a satellite in constant sunlight, for instance.

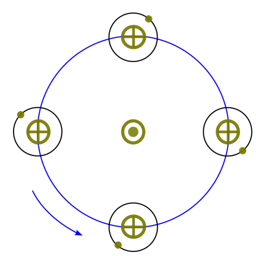

Figure 6.3: Sun-Sync.

#### Molniya Orbit

According to Eq. (6.19b), $\langle\dot{\omega}\rangle = 0$ when $5\cos^2(i) - 1 = 0$, which is the case for $i \approx 63.4°$ or $i \approx 116.6°$. An orbit with one of these two inclinations is known as a "frozen orbit", since its argument of perigee does not rotate over time and its perigee and apogee are fixed relative to Earth. When such an orbit is also highly elliptical, the resulting Molniya orbit can be used for high latitude communication, since a spacecraft in such an orbit spends the majority of its time close to the fixed apogee, as shown in Figure 6.4.

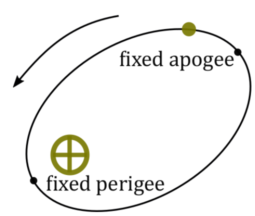


Figure 6.4: Molniya