Lecture 7

Launch Vehicle Dynamics

SOME of the basic considerations and governing relationships of launch vehicles and rocket dynamics are presented, sufficient only for a brief introduction to the topic. Simplifying assumptions pertinent to each stage of a typical launch trajectory are made, and previous dynamics and orbital mechanics concepts are used to study the mathematics and physics of each stage.

Overview

Today's launch vehicles typically consist of multiple stages, each capable of providing thrust and/or carrying a payload. At various predetermined points on the vehicle's trajectory, each lower stage is jettisoned upon having most (if not all) of its fuel consumed. As a result of the mass reduction that ensues from jettisoning each lower stage, the remaining part of the vehicle can perform much more efficiently.

In general, a launch trajectory may consist of the following stages:

- Vertical Ascent: initial path that is kept perpendicular to Earth's surface in order to minimize aerodynamic heating and enable exiting the dense lower atmosphere as quickly as possible
- Turn-Over Trajectory: controlled tilting of the flight path to achieve a desired horizontal velocity
- Gravity Turn Trajectory: gradual transition to near-horizontal flight, primarily driven by Earth' gravitational force

Vertical Ascent

Consider and Earth-fixed inertial frame, $\mathscr{F}_I$, with origin O_I (at Earth's centre, for example). Because all the forces, namely weight, $\underset{\rightarrow}{\boldsymbol{W}}$; propulsion thrust, $\underset{\rightarrow}{\boldsymbol{T}}$; and aerodynamic drag, $\underset{\rightarrow}{\boldsymbol{D}}$, act in the vertical direction as shown in Figure 7.1, the motion equation is one-dimensional in this phase:

$$m\underset{\rightarrow}{\boldsymbol{r}}^{\bullet\bullet} = \underset{\rightarrow}{\boldsymbol{f}} \quad \Rightarrow \quad m\dot{v} = T - D - W \tag{7.1}$$

where m, $\underset{\rightarrow}{r}$, and v are the spacecraft's mass and centre of mass position (relative to O_I) and speed, respectively. The magnitudes of the forces are given by:

$$T = -v_e\dot{m} + (\Delta P)A_e \approx -v_e\dot{m} = -(g_0 I_{sp})\dot{m} \;\;,\;\; W = mg(t) \;\;,\;\; D = \frac{1}{2}\rho v^2 C_D S \tag{7.2}$$

where v_e, ΔP and A_e are the exhaust speed, pressure difference, and exit area associated with the rocket's nozzle, and $\dot{m}$ (a negative flow rate) denotes the rate of fuel consumption. The rocket's specific impulse, $I_{sp} \triangleq v_e/g_0$, is a measure of its performance, where $g_0 = 9.81\ \mathrm{m/s^2}$ is the gravitational acceleration at the sea level. Note, however, that $g(t)$ is, strictly speaking, a function of time as it depends on the rocket's altitude. Lastly, S and C_D are the reference area and drag coefficient (a function of speed and altitude) associated with rocket's movement through air, and ρ (also a function of altitude) is the atmospheric density.

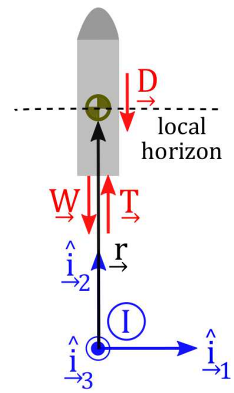

Figure 7.1: Vertical Ascent

We neglect drag, keeping in mind that its presence will result in about 10% decrease in performance. Assuming constant fuel consumption, we can integrate Eq. (7.1) (upon substituting Eq. (7.2) in it) as follows:

$$m\dot{v} \approx -g_0 I_{sp}\dot{m} - mg \;\;\Rightarrow\;\; \int_{\cancel{v_0}\,0}^{v} dv = -g_0 I_{sp}\int_{m_0}^{m}\frac{dm}{m} - \int_0^t g(t)\,dt \;\;\Rightarrow\;\; v(t) \approx -g_0 I_{sp}\ln\left(\frac{m}{m_0}\right) - g_0 t \tag{7.3}$$

where another approximation is made to set $g(t) \approx g_0$, noting that the difference is within 10% up to an altitude of 300 km. Integrating Eq. (7.3) again with respect to time yields an estimate of the rocket's altitude:

$$h(t) \triangleq y(t) - y_0 \approx -g_0 I_{sp}\int_0^t \ln\left(\frac{m}{m_0}\right) dt - \frac{1}{2}g_0 t^2 \tag{7.4}$$

where y_0 is the rocket's initial (at lift-off) 2-component, depending on the reference frame selected. Further simplification results from using the mass flow relationship:

$$m(t) = m_0 + \dot{m}t \;\;\Rightarrow\;\; \int_0^t \ln\left(\frac{m}{m_0}\right) dt = \int_0^t \ln(m)\,dt - \int_0^t \ln(m_0)\,dt = \frac{1}{\dot{m}}\Big[m\ln(m) - m\Big]_{m_0}^{m} - t\ln(m_0) \tag{7.5}$$

where $dt = (1/\dot{m})\,dm$ is used. Upon substituting Eq. (7.5) back into Eq. (7.4) we obtain:

$$h(t) \approx g_0 I_{sp}\left[1 - \frac{1}{(m_0/m) - 1}\ln\left(\frac{m_0}{m}\right)\right] t - \frac{1}{2}g_0 t^2 \tag{7.6}$$

where, once again, $m(t) = m_0 + \dot{m}t$ is used.

Turn-Over Trajectory

"Thrust-vectoring" is achieved (via gimballed nozzles, for example) to produce a thrust vector with an angle of $\delta(t)$ relative to the vehicle's roll axis. To avoid high transverse loads on the structure, δ should be kept small. As shown in Figure 7.2, let the vehicle's angle of attack (from its velocity vector to its roll axis) and *flight path* *angle* (from local horizontal to its velocity vector) be represented by $\alpha(t)$ and $\gamma(t)$, respectively, and define the vehicle's pitch angle as $\theta \triangleq \gamma + \alpha$.

Note: The flight path angle changes over time owing to two factors:

- change in pitch as a result of the vehicle's rotational dynamics, producing $\Delta_1\gamma(t) = \int_0^t \dot{\theta}\, dt$
- change in local horizon due to Earth's curvature as the vehicle deviates $dx \approx (R_\oplus + h)\, d\gamma$ (or, if O_I is at Earth's centre, $dx \approx y\, d\gamma$) from its initially vertical path, producing $\Delta_2\gamma(t) = \int_0^x 1/(R_\oplus + h)\, dx$

Similarly to Eq. (7.2), the magnitudes of the forces acting on the vehicle in this stage are:

$$T \approx -(g_0 I_{sp})\dot{m} \;\;,\;\; W = mg(t) \;\;,\;\; D = \frac{1}{2}\rho v^2 C_D S \;\;,\;\; L = \frac{1}{2}\rho v^2 C_L S \tag{7.7}$$

where L and C_L represent the lift force and coefficient, respectively, and both C_D and C_L are now also functions of the angle of attack, α. Both $\underset{\rightarrow}{\boldsymbol{L}}$ and $\underset{\rightarrow}{\boldsymbol{D}}$ are taken to act through the centre of pressure, with a distance of b from the vehicle's centre of mass, while $\underset{\rightarrow}{\boldsymbol{T}}$ is applied at a distance of a from the centre of mass.

In addition to the inertial frame, $\mathscr{F}_I$, introduced earlier, let us define $\mathscr{F}_V$ and $\mathscr{F}_B$, both with their origin fixed to the rocket and their 3-axis aligned with $\hat{\underset{\rightarrow}{\boldsymbol{i}}}_3$, such that $\hat{\underset{\rightarrow}{\boldsymbol{v}}}_1$ is parallel to the velocity vector, $\underset{\rightarrow}{\boldsymbol{v}}$, and $\hat{\underset{\rightarrow}{\boldsymbol{b}}}_1$ is aligned with the vehicle's roll axis. The translational equations of planar motion are then given by:

$$m\underset{\rightarrow}{\boldsymbol{r}}^{\bullet\bullet} = \underset{\rightarrow}{\boldsymbol{f}} \;\;\Rightarrow\;\; m\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix} = \begin{bmatrix}0\\ -mg\\ 0\end{bmatrix} + \boldsymbol{C}_{IV}\begin{bmatrix}-D\\ L\\ 0\end{bmatrix} + \boldsymbol{C}_{IB}\begin{bmatrix}T\cos(\delta)\\ T\sin(\delta)\\ 0\end{bmatrix} \tag{7.8}$$

where $\boldsymbol{C}_{IV} = \boldsymbol{C}_3^{\mathsf{T}}(\gamma - \Delta_2\gamma)$ and $\boldsymbol{C}_{IB} = \boldsymbol{C}_3^{\mathsf{T}}(\theta - \Delta_2\gamma)$ are the rotation matrices from $\mathscr{F}_V$ and $\mathscr{F}_B$, respectively, to $\mathscr{F}_I$. The rocket's attitude is represented by $\boldsymbol{C}_{BI}$.

The rotational equation of motion (from DYNAMICS) for pitch can be written as:

$$\boldsymbol{I}\overset{\bullet}{\boldsymbol{\omega}} + \boldsymbol{\omega}^\times \boldsymbol{I}\boldsymbol{\omega} = \boldsymbol{\tau}_{\oplus} \;\;\Rightarrow\;\; I_3\ddot{\theta} = \tau_3 = \big[L\cos(\alpha) + D\sin(\alpha)\big]b - T\sin(\delta)a \tag{7.9}$$

where I_3 is the third principal moment of inertia (corresponding to pitch), and τ_3 is the only non-zero component (out-of-plane) of the external torque on the vehicle about its centre of mass. Note that $N \triangleq L\cos(\alpha) + D\sin(\alpha)$ is the transverse force (normal to $\underset{\rightarrow}{\boldsymbol{v}}$) applied at the centre of pressure.

The desired motion of the rocket can be achieved by controlling α and δ accordingly. For example, for a constant rate turn-over manoeuvre with $\ddot{\theta} = 0$, thrust vectoring of $\delta = \sin^{-1}\big(Nb/(Ta)\big)$ should be used.

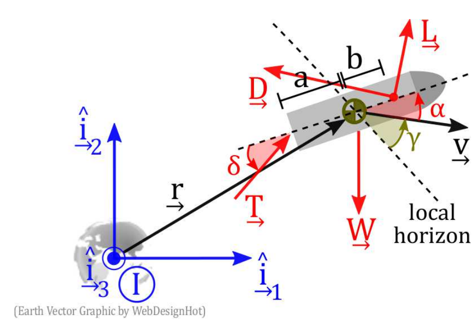

Figure 7.2: Forces and Angles involved in Turn-Over Manoeuvre

Gravity Turn Trajectory

Transition to horizontal flight is achieved in this phase by gradual reduction of γ to zero. In order to avoid a large angle of attack and thrust vector angle because of the associated aerodynamic heating and loading issues, this may be achieved by setting both $\delta = 0$ and $\alpha = 0$, hence maintaining $\underset{\rightarrow}{\boldsymbol{T}}$ parallel to $\underset{\rightarrow}{\boldsymbol{v}}$. This implies $\mathscr{F}_V$ and $\mathscr{F}_B$ are aligned and $\theta = \gamma$, hence reducing Eq. (7.8):

$$m\underset{\rightarrow}{\boldsymbol{r}}^{\bullet\bullet} = \underset{\rightarrow}{\boldsymbol{f}} \quad \Rightarrow \quad m \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ -mg \\ 0 \end{bmatrix} + \boldsymbol{C}_3^{\mathsf{T}}(\gamma - \Delta_2\gamma) \begin{bmatrix} -\cancelto{\approx 0}{D} + T \\ \cancelto{\approx 0}{L} \\ 0 \end{bmatrix} \tag{7.10}$$

and noting that $\sin(\alpha) = \sin(\delta) = 0$ and that negligibly small lift and drag are generated owing to the vehicle's zero angle of attack, Eq. (7.9) implies $\ddot{\theta} = 0$, so $\dot{\gamma}$ is constant. However, the angle relative to the inertial frame (measured from $\underset{\rightarrow}{\hat{\boldsymbol{i}}}_3$, for example) does not necessarily have a constant rate because of the time-variation of the local horizontal due to Earth's curvature.

Orbit Injection

Assuming the conditions after burnout (termination of thrust) are known, the approach discussed in ORBIT DESCRIPTION AND DETERMINATION can be used to determine the spacecraft's resulting orbital parameters:

Given $\underset{\rightarrow}{\boldsymbol{r}}_o$ and $\underset{\rightarrow}{\boldsymbol{v}}_o$ (or their magnitudes, r_o and v_o, and the flight path angle, γ_o), one can compute:

1. specific angular momentum (or its magnitude):

$$\underset{\rightarrow}{\boldsymbol{h}} = \underset{\rightarrow}{\boldsymbol{r}}_o \times \underset{\rightarrow}{\boldsymbol{v}}_o \quad , \quad h = r_o v_o \sin\left(\frac{\pi}{2} - \gamma_o\right) = r_o v_o \cos(\gamma_o)$$

2. specific energy and semi-major axis:

$$\epsilon = \frac{v_o^2}{2} - \frac{\mu_\oplus}{r_o} \quad , \quad a = \frac{-\mu_\oplus}{2\epsilon}$$

3. eccentricity vector (or its magnitude):

$$\underset{\rightarrow}{\boldsymbol{e}} = \frac{\underset{\rightarrow}{\boldsymbol{v}}_o \times \underset{\rightarrow}{\boldsymbol{h}}}{\mu_\oplus} - \frac{\underset{\rightarrow}{\boldsymbol{r}}_o}{r_o} \quad , \quad e = \sqrt{1 + \frac{2\epsilon h^2}{\mu_\oplus^2}}$$

4. spacecraft's initial true anomaly in the orbit:

$$r_o = \frac{h^2/\mu_\oplus}{1 + e\cos(\theta_o)} \quad \Rightarrow \quad \theta_o = \cos^{-1}\left[\frac{1}{e}\left(\frac{h^2}{\mu_\oplus r_o} - 1\right)\right]$$

Any subsequent adjustments to the orbit can then follow using thrusters (or other actuators). Such orbital manoeuvres will be studied next.