Lecture 9

Interplanetary Travel

EQUIPPED with an understanding of orbital motion and how to modify spacecraft's orbits, we can now tackle the problem of travelling from one planet to another. This usually involves at least three main stages: (1) escaping the home planet's orbit, (2) orbiting about Sun towards the target planet, and (3) entering an orbit about the target planet.

Overview

Suppose we would like to launch a spacecraft from Earth, $\oplus$, to Jupiter, ♃, both assumed to have circular orbits as shown in Figure 9.1. In theory, we have an n-body problem in which many celestial bodies simultaneously exert gravitational forces on the spacecraft. In order to simplify the analysis and for preliminary mission design purposes, the so-called "patched conic approximation" (PCA) may be used, in which:

- if the spacecraft is within the r_{SOI} of a planet ($\oplus$ or ♃, or other planets encountered en-route), it is considered to be in 2-body motion with that planet only.
- if the spacecraft is outside the r_{SOI} of a planet, it is considered to be in 2-body motion with Sun.

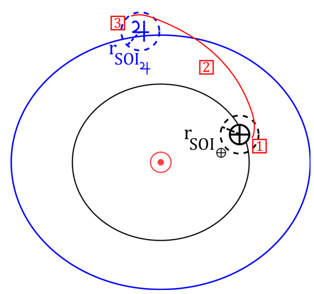


Figure 9.1: Overview

where r_{SOI} refers to the radius of the planet's *sphere of influence* (SOI).

A Criterion for Sphere of Influence

The definition of r_{SOI} is somewhat arbitrary, as gravitational forces of external bodies still exist in the proximity of the SOI boundary; however, a consideration of the relative effects of the planet's gravity on the spacecraft's acceleration compared to that of Sun provides a reasonable criterion for r_{SOI}.

As shown in Figure 9.2, the spacecraft's motion about Sun is described by:

$$\cancel{m_c}\ddot{\underset{\rightarrow}{\boldsymbol{r}}}_c = \underset{\rightarrow}{\boldsymbol{f}}_{c\odot} + \underset{\rightarrow}{\boldsymbol{f}}_{cp} = -\frac{Gm_\odot \cancel{m_c}}{r_c^3}\underset{\rightarrow}{\boldsymbol{r}}_c - \frac{Gm_p\cancel{m_c}}{\rho^3}\underset{\rightarrow}{\boldsymbol{\rho}} \;\Rightarrow\; \ddot{\underset{\rightarrow}{\boldsymbol{r}}}_c = -\frac{Gm_\odot}{r_c^3}\underset{\rightarrow}{\boldsymbol{r}}_c - \frac{Gm_p}{\rho^3}\underset{\rightarrow}{\boldsymbol{\rho}} \tag{9.1}$$

where $\boldsymbol{f}_{c\odot}$ and $\boldsymbol{f}_{cp}$ are the forces on the spacecraft exerted by Sun and the planet, and m_c, m_p, and $m_\odot$ denote the mass of the spacecraft, the planet, and Sun, respectively. The position vectors $\boldsymbol{r}_c$ and $\boldsymbol{r}_p$ represent the spacecraft's and the planet's absolute position relative to Sun, while $\boldsymbol{\rho}$ is relative position of the spacecraft from the planet. Letting $\boldsymbol{A}_\odot \triangleq -(Gm_\odot/r_c^3)\boldsymbol{r}_c$ and $\boldsymbol{A}_p \triangleq (Gm_p/\rho^3)\boldsymbol{\rho}$ represent the spacecraft's acceleration (relative to Sun) due to Sun and the planet, respectively, we find the ratio of the magnitudes of these two parameters, assuming $1/r_c^2 \approx 1/r_p^2$:

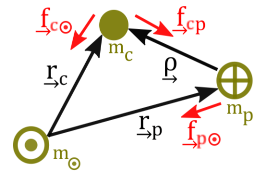


Figure 9.2: Force and Position Vectors

$$A_\odot = \frac{Gm_\odot}{r_c^2} \approx \frac{Gm_\odot}{r_p^2} \quad , \quad A_p = \frac{Gm_p}{\rho^2} \quad \Rightarrow \quad \frac{A_p}{A_\odot} \approx \frac{m_p}{m_\odot}\left(\frac{r_p}{\rho}\right)^2 \tag{9.2}$$

which is a relative measure of the planet's influence on the spacecraft's orbit about Sun.

The spacecraft's motion about the planet is described by:

$$\ddot{\boldsymbol{\rho}} = \ddot{\boldsymbol{r}}_c - \ddot{\boldsymbol{r}}_p = \ddot{\boldsymbol{r}}_c - \frac{\boldsymbol{f}_{p\odot}}{m_p} = \left(-\frac{Gm_\odot}{r_c^3}\boldsymbol{r}_c - \frac{Gm_p}{\rho^3}\boldsymbol{\rho}\right) + \left(\frac{Gm_\odot}{r_p^3}\boldsymbol{r}_p\right) \tag{9.3}$$

where $\boldsymbol{f}_{p\odot}$ is Sun's gravitational force on the planet. Rearranging Eq. (9.3) and assuming $1/r_c^3 \approx 1/r_p^3$ yields:

$$\ddot{\boldsymbol{\rho}} \approx -\frac{Gm_\odot}{r_p^3}\cancelto{\boldsymbol{\rho}}{(\boldsymbol{r}_c - \boldsymbol{r}_p)} - \frac{Gm_p}{\rho^3}\boldsymbol{\rho} \tag{9.4}$$

Defining $\boldsymbol{a}_\odot \triangleq -(Gm_\odot/r_p^3)\boldsymbol{\rho}$ and $\boldsymbol{a}_p \triangleq -(Gm_p/\rho^3)\boldsymbol{\rho}$ as the spacecraft's acceleration (relative to the planet) due to Sun and the planet, respectively, we can take the ratio of the magnitudes of the two contributions:

$$a_\odot = \frac{Gm_\odot}{r_p^3}\rho \quad , \quad a_p = \frac{Gm_p}{\rho^2} \quad \Rightarrow \quad \frac{a_\odot}{a_p} \approx \frac{m_\odot}{m_p}\left(\frac{\rho}{r_p}\right)^3 \tag{9.5}$$

which is a relative measure of Sun's influence on the spacecraft's orbit about the planet.

We now take the SOI to be the region in which the gravitational effects of the planet dominate over those of the Sun, mathematically represented by:

$$\frac{a_\odot}{a_p} < \frac{A_p}{A_\odot} \quad \Rightarrow \frac{m_\odot}{m_p}\left(\frac{\rho}{r_p}\right)^3 < \frac{m_p}{m_\odot}\left(\frac{r_p}{\rho}\right)^2 \quad \Rightarrow \quad r_{SOI} = r_p\left(\frac{m_p}{m_\odot}\right)^{2/5} \tag{9.6}$$

where r_{SOI} is taken as the lower bound of the comparison inequality.

Stage 2 - Interplanetary Hohmann Transfer

Assume the elliptic transfer orbit between the two planets' orbits is going to be a Hohmann transfer, although it does not have to be. In this case, the transfer ellipse is tangent to both planets' orbits, so for a mission from $\oplus$ to ♃, the spacecraft's velocity vector (relative to Sun, $\odot$) upon departure (exiting the SOI of $\oplus$) should be parallel to $\boldsymbol{v}_\oplus$. The SOI is assumed to be so small in the interplanetary scale that the position of the spacecraft (relative to Sun, $\odot$) upon departure is taken to be the same as that of $\oplus$. From ORBITAL MANOEUVRES,

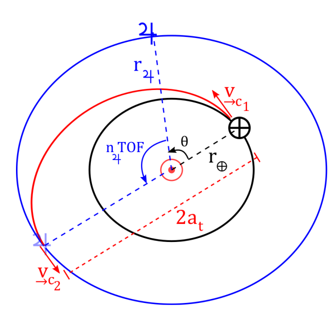


Figure 9.3: Interplanetary Hohmann Transfer

for the Hohmann transfer, an elliptic orbit about $\odot$, we have:

$$a_t = \frac{r_\oplus + r_{♃}}{2} \quad , \quad TOF = \pi\sqrt{\frac{a_t^3}{\mu_\odot}} \tag{9.7}$$

where $r_\oplus$ and $r_{♃}$ are the radii of the circular orbits of $\oplus$ and ♃ about $\odot$. But we need ♃ to meet the spacecraft when it arrives at its orbit, so we compute the relative phase angle between the two planets, θ, at the departure time as follows:

$$n_\oplus = \frac{2\pi}{T_\oplus} \quad , \quad n_{♃} = \frac{2\pi}{T_{♃}} \quad , \quad \theta = \pi - n_{♃} \cdot TOF \tag{9.8}$$

where $n_\oplus$ and $n_{♃}$ are the mean motions of the planets, in $\mathrm{rad/s}$, about $\odot$. The term $(n_{♃} \cdot TOF)$ describes how much ♃ travels during the interplanetary travel time. The phase angle, θ, determines when a "launch window" exists. The time between successive launch windows is known as a *synodic period*, and is given by:

$$T_{syn} = \frac{2\pi}{|n_{♃} - n_\oplus|} \tag{9.9}$$

where the denominator provides the relative angular rate of motion of $\oplus$ with respect to ♃, so that the two planets reach the same relative position, once again, after T_{syn}.

Based on the vis-viva equation from ORBITAL MECHANICS, the spacecraft's heliocentric speed upon exiting the SOI of $\oplus$, v_{c_1}, and that upon entering the SOI of ♃, v_{c_2}, are given by:

$$v_{c_1} = \sqrt{\mu_\odot\left(\frac{2}{r_\oplus} - \frac{1}{a_t}\right)} \quad , \quad v_{c_2} = \sqrt{\mu_\odot\left(\frac{2}{r_{♃}} - \frac{1}{a_t}\right)} \tag{9.10}$$

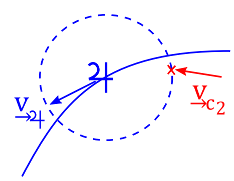


Figure 9.4: Arrival

Note: In general, $\underset{\to}{v}_{c_2}$ need not be parallel to $\underset{\to}{v}_{♃}$, as shown in Figure 9.4.

Patch Conditions

For PCA, the mathematical relationships at the boundaries of the SOI's involved - known as the "patch conditions" - are (as shown in Figure 9.2):

$$\underset{\to}{r}_c = \underset{\to}{r}_p + \underset{\to}{\rho} \tag{9.11a}$$

$$\underset{\to}{v}_c = \underset{\to}{v}_p + \underset{\to}{\nu} \tag{9.11b}$$

where $\underset{\to}{r}$ and $\underset{\to}{v}$ are position and velocity vectors with respect to Sun, $\odot$, while $\underset{\to}{\rho}$ and $\underset{\to}{\nu}$ are those with respect to the planet, $\oplus$ or ♃ in our example.

Stage 1 - Departure Trajectory

As illustrated in Figure 9.5, this stage usually involves a parabolic or hyperbolic trajectory (both of which have $\epsilon \geqslant 0$, as required) about the home planet, $\oplus$ in our case. Recall that the interplanetary Hohmann transfer is to be tangent to the circular orbit of $\oplus$, so we need $\underset{\to}{v}_{c_1}$ parallel to $\underset{\to}{v}_\oplus$ (both heliocentric) upon exiting the SOI of $\oplus$. To achieve this goal, we align the asymptote of the departure hyperbola to be parallel to $\underset{\to}{v}_\oplus$.

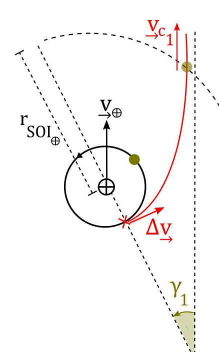


Figure 9.5: Heliocentric

Note: In this section, the subscript '1' is used to denote a departure-related quantity.

Rearranging the patch condition in Eq. (9.11b), applied to the SOI boundary of $\oplus$, and using magnitudes only (since $\underset{\to}{v}_{c_1}$ and $\underset{\to}{v}_{\oplus}$ are parallel) provides the hyperbolic excess speed, $\underset{\to}{\nu}_{\infty}$:

$$\nu_1 = v_{c_1} - v_{\oplus} = \nu_{\infty_1} \tag{9.12}$$

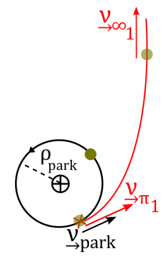


Figure 9.6: Geocentric

where v_{c_1} is given by the left-hand side relationship in Eq. (9.10), since it is the spacecraft's absolute speed once exiting the SOI of $\oplus$ and entering the Hohmann transfer. To find $\Delta\nu_{dep}$ of departure, we need to know $\underset{\to}{\nu}_{\pi_1}$, the geocentric velocity that the spacecraft will require upon embarking on its hyperbolic trajectory from its parking orbit about $\oplus$, shown in Figure 9.6. In other words, it is the spacecraft's speed at the periapsis of the hyperbola. In order to successfully apply the vis-viva equation at the periapsis to find ν_{dep}, we need to determine the semi-major axis, a_1, of the hyperbolic departure trajectory (which is negative for a hyperbola). To this end, the orbit's constant specific energy can be used:

$$\epsilon_1 = \frac{-\mu_{\oplus}}{2a_1} = \lim_{\rho_1\to\infty}\left(\frac{\nu_1^2}{2} - \frac{\mu_{\oplus}}{\rho_1}\right) = \frac{\nu_{\infty_1}^2}{2} \quad\Rightarrow\quad a_1 = \frac{-\mu_{\oplus}}{\nu_{\infty_1}^2} \tag{9.13}$$

where ρ_1 denotes the spacecraft's radial distance from $\oplus$ on its hyperbolic trajectory. With a_1 determined, the via-viva equation, applied both pre- and post-thrust at the manoeuvre node, can be used:

$$\Delta\nu_1 = \nu_{\pi_1} - \nu_{park} = \sqrt{\mu_{\oplus}\left(\frac{2}{\rho_{park}} - \frac{1}{a_1}\right)} - \sqrt{\frac{\mu_{\oplus}}{\rho_{park}}} \tag{9.14}$$

where ρ_{park} is both the radius of the parking orbit (assumed to be circular) and the periapsis distance of the hyperbolic departure trajectory. Provided by Eq. (9.14) is how much $\Delta\nu$ will be needed to initiate the hyperbolic escape, but at which location on the parking orbit should this thrust be applied? To answer this question, the geometry and orientation of the hyperbola, as shown in Figure 9.7, should be considered.

The polar equation with $\theta = 0$ relates the periapsis distance to eccentricity:

$$\rho_{park} = \frac{a_1(1-e_1^2)}{1+e_1\cancelto{1}{\cos(\theta)}} = a_1(1-e_1) \quad\Rightarrow\quad e_1 = 1 - \frac{\rho_{park}}{a_1} \tag{9.15}$$

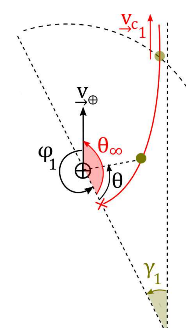


Figure 9.7: Phase

In addition, as mentioned in ORBITAL MECHANICS, using the polar equation with $\rho_1 \to \infty$ yields the escape true anomaly:

$$1 + e_1\cos(\theta_{\infty_1}) = 0 \quad\Rightarrow\quad \theta_{\infty_1} = \cos^{-1}\left(\frac{-1}{e_1}\right) \tag{9.16}$$

which directly relates to the half-angle of the departure hyperbola, γ_1:

$$\gamma_1 + \theta_{\infty_1} = \pi \quad\Rightarrow\quad \cos(\theta_{\infty_1}) = \cos(\pi-\gamma_1) = -\cos(\gamma_1) = \frac{-1}{e_1} \quad\Rightarrow\quad \gamma_1 = \cos^{-1}\left(\frac{1}{e_1}\right) \tag{9.17}$$

Therefore, the required phase angle between the manoeuvre node and the velocity vector of $\oplus$ relative to $\odot$, $\underset{\to}{v}_{\oplus}$, is given by:

$$\phi_1 = \pi + \gamma_1 = \pi + \cos^{-1}\left(\frac{1}{e_1}\right) \quad or \quad \phi_1 = \gamma_1 = \cos^{-1}\left(\frac{1}{e_1}\right) \tag{9.18}$$

where the left-hand side relationship is for the case in which the target planet has a larger orbit than the home planet, while the one of the right-hand side is for a transfer to a smaller orbit (in which case $\underrightarrow{\boldsymbol{\nu}}_\infty$ would be in the opposite direction of $\underrightarrow{\boldsymbol{v}}_\oplus$, and ν_∞ from Eq. (9.12) would be negative).

Stage 3 - Arrival Trajectory

For arrival, as shown in Figure 9.8, the procedure is reversed. Rearranging the patch conditions in Eq. (9.11) applied to the SOI boundary of ♃yields:

$$\underrightarrow{\boldsymbol{\rho}}_2 = \underrightarrow{\boldsymbol{r}}_{c_2} - \underrightarrow{\boldsymbol{r}}_{♃} \tag{9.19a}$$

$$\underrightarrow{\boldsymbol{\nu}}_{\infty_2} = \underrightarrow{\boldsymbol{v}}_{c_2} - \underrightarrow{\boldsymbol{v}}_{♃} \tag{9.19b}$$

where v_{c_2} is given by the right-hand side relationship in Eq. (9.10), since it is the spacecraft's absolute speed upon exiting the Hohmann transfer and entering the SOI of ♃. If the SOI is small enough, $\underrightarrow{\boldsymbol{v}}_{c_2}$ and $\underrightarrow{\boldsymbol{v}}_{♃}$ should be nearly parallel, but in order to avoid collision, a non-zero parameter b_2 (with $-b_2$ being equal to the perpendicular distance of the focus from the asymptotes) is required.

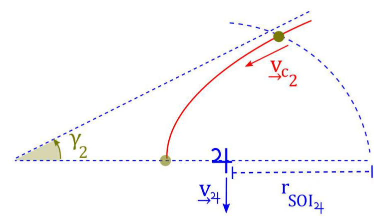


Figure 9.8: Heliocentric

Similar to Eq. (9.13) from departure, we can find a_2, and subsequently use trigonometry to find b_2 of the arrival hyperbola:

$$a_2 = \frac{-\mu_{♃}}{\nu_{\infty_2}^2} \quad , \quad b_2 = -\rho_{\infty_2}\sin(\psi) = -r_{SOI}\sin\left[\cos^{-1}\left(\frac{-\underrightarrow{\boldsymbol{\rho}}_{\infty_2}\cdot\underrightarrow{\boldsymbol{\nu}}_{\infty_2}}{\rho_{\infty_2}\nu_{\infty_2}}\right)\right] \tag{9.20}$$

where ψ, shown in Figure 9.9, is the angle between the relative arrival velocity of the spacecraft, $\underrightarrow{\boldsymbol{\nu}}_{\infty_2}$, and the its relative position upon entering the SOI of ♃, $\underrightarrow{\boldsymbol{\rho}}_{\infty_2}$ (the magnitude of which equals r_{SOI}).

Having determined a and b, the definition of the parameter b for a hyperbolic orbit can be used to find the arrival trajectory's eccentricity:

$$b_2 \triangleq a_2\sqrt{e_2^2 - 1} \quad \Rightarrow \quad e_2 = \sqrt{1 + \frac{b_2^2}{a_2^2}} \tag{9.21}$$

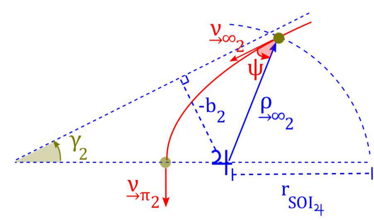


Figure 9.9: Joviocentric

Assuming a capture orbit of radius ρ_{cap} is desired, such that $\rho_{cap} > R_{♃}$ (radius of ♃) to avoid collision, the following holds analogously to the left-hand side relationship in Eq. (9.15):

$$\rho_{cap} = a_2(1 - e_2) \tag{9.22}$$

which is also the periapsis distance of the arrival hyperbola. Finally, using the vis-viva equation again, we obtain:

$$\Delta\nu_2 = \nu_{\pi_2} - \nu_{cap} = \sqrt{\mu_{♃}\left(\frac{2}{\rho_{cap}} - \frac{1}{a_2}\right)} - \sqrt{\frac{\mu_{♃}}{\rho_{cap}}} \tag{9.23}$$

where ρ_{cap} is both the radius of the capture orbit (assumed to be circular) and the periapsis distance of the hyperbolic arrival trajectory. It should be noted that ρ_{cap} can be controlled by modifying the b_2 parameter of the hyperbola via manipulating the entry point to the SOI; or by changing the spacecraft's arrival velocity, which typically requires more energy and fuel.

General (Non-Hohmann) Interplanetary Trajectory

The desired trajectory between the home and the target planets cannot always be a Hohmann transfer, especially if a short TOF is desired or required. In that case, Lambert's problem can be solved, using the home planet's position at the departure time, $\vec{r}_1 = \vec{r}_\oplus(t_1)$, as the initial position vector; the target planet's position at the arrival time, $\vec{r}_2 = \vec{r}(t_2)$, as the final position vector; and the desired mission duration as the TOF between the two position vectors. The approach described in ORBIT DESCRIPTION AND DETERMINATION can then be used to obtain the spacecraft's heliocentric velocity vector at the beginning of the interplanetary transfer, $\vec{v}_1$, and its final velocity vector at the end of it, $\vec{v}_2$. Alternative methods of solving Lambert's problem could also be used, for example if the phase angle is too larger than $90°$.

The calculations corresponding to Stages 1 and 3 remain very similar to those described above for a Hohmann transfer, except the departure velocity vector no longer has to be parallel to the planet's velocity vector. However, $\vec{\nu}_1 = \vec{v}_{c_1} - \vec{v}_\oplus = \vec{\nu}_{\infty_1}$ is still valid in vectorial form, and can be used to obtain $\nu_1 = |\vec{\nu}_1|$. Similar comments hold for the arrival excess velocity.

Planetary Fly-By

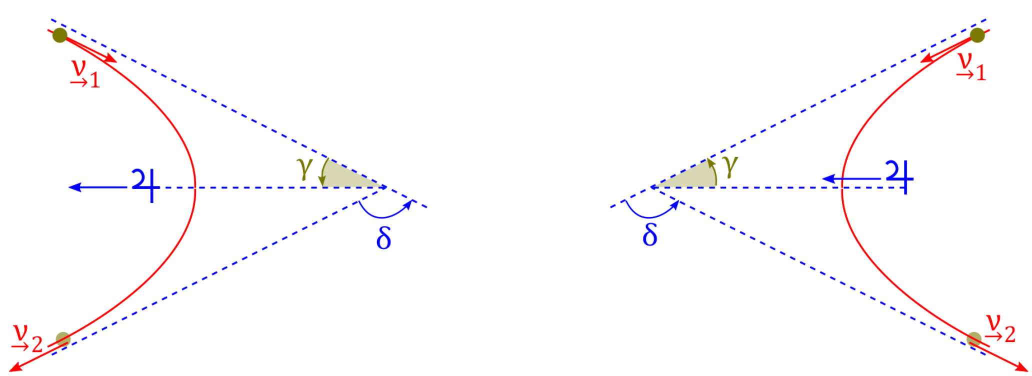
Also known as "gravity assist" and "gravity braking", these phenomena result in an increase or decrease in the spacecraft's velocity vector as a result of the gravitational force of the fly-by body. Consider the fly-by arrival and departure velocities relative to the planet (♃, for example), $\vec{\nu}_1$ and $\vec{\nu}_2$, and the corresponding heliocentric velocities, $\vec{v}_1$ and $\vec{v}_2$.

For a *trailing* fly-by, illustrated in Figure 9.10a, $\nu_1 = \nu_2$, but $\vec{\nu}_1$ and $\vec{\nu}_2$ are not parallel. We have:

$$\vec{v}_1 = \vec{v}_{♃} + \vec{\nu}_1 \quad , \quad \vec{v}_2 = \vec{v}_{♃} + \vec{\nu}_2 \quad \Rightarrow \quad \vec{v}_2 > \vec{v}_1 \tag{9.24}$$

which implies that the planet's deflecting the motion of the spacecraft through an angle of $\delta = \pi - 2\gamma$ results in an increase in the spacecraft's absolute speed. In other words, $\Delta\epsilon = (v_2^2 - v_1^2)/2 > 0$ for the heliocentric transfer orbit, which implies that the planet exerts $\Delta\vec{v}$ on the spacecraft. This type of fly-by is useful in mission design for a fuel-free increase in the travel distance.

Similarly, for a *leading* fly-by, shown in Figure 9.10b, an opposite effect results in $\vec{v}_2 < \vec{v}_1$ and $\Delta\epsilon < 0$. This fly-by serves well for braking purposes, with the goal of transferring to a smaller orbit, for instance.



(a) Trailing Fly-By (b) Leading Fly-By

Figure 9.10: Change of Spacecraft's Relative Velocity during Planetary Fly-By